The $\frac{2}{3}$-Černý Conjecture and synchronization on finite state automata

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Abstract. Synchronization is an important topic in theoretical computer science, and it is also important in the theory of finite state automata. Deterministic finite state automata that are synchronizing are interesting examples of error resilient systems: if a synchronizing automaton is taken out of control, it can be reset to a specific state by feeding it with a synchronizing string.The study of minimal synchronizing strings is an old topic of Automata Theory and the classical Černý conjecture refers to it. The later conjecture states that the minimal reset length of any n- state synchronizing automata is bounded above by $(n-1)^2$. This work is an attempt to understand the main questions behind Černý conjecture. We study the synchronizing time of triples. We conjecture that the time to synchronizing triples is bounded above $\frac{2}{3}n^2$. We call this conjecture $\frac{2}{3}$ - Černý conjecture. We discuss the meaning and relevance of our conjecture and we study the case of eulerian automata and get some partial results.

1. Introduction

The concept of synchronizing automata appears for the first time in the work of Moore [1]. A finite state automaton is an elementary model of devices that work in discrete time, such as computers or relay control systems . This leads to a natural question: is it possible to restore the control of such device, if the current state is not known, but the outputs produced by the device can be observed under several actions? Moore proved that under certain conditions it is possible to determine the state to which the automaton will arrive after an adequate sequence of actions called experiments. The Gedanken-experiments of Moore motivated the study of synchronizing automata which is nowadays, an active research field. The recent review of Sandberg [2] is a good reference in the area.

The study of automation processes has been an important area of research in Colombia. Research and development of automation processes is a vital factor in the development of the manufacturing sector [3]. The university Francisco de Paula Santander Ocaña has been a pioneer in the study of automation in the Norte del Santander department [4].

Jan Černý [5] published his first paper on automata synchronization in 1964. He presented in this paper a sequence that we denote with the symbol $\{C_n\}_{n\geq 2}$. Automaton C_n is a $n-$state automaton that can be synchronized by a word of length $(n-1)^2$. Černý conjectured, that every synchronizing automaton has a synchronizing word whose length is upperbounded by $(n-1)^2$. This conjecture is considered as the most important open problem in the combinatorial theory of finite automata [6] .

A first attempt to solve the Černý problem consists in reduce its proof to a small and representative enough class of automata. Volkov [7] proved the following result in that direction:

Suppose that Černý conjecture holds for all complete automata with a strongly connected transition graph. Then the conjecture holds for all complete automata, both strongly connected and otherwise.
The above result motivated the study of different classes of automata for which Černý conjecture holds true. Let us list some of the most important results in this direction.
1. **Circular automata**: an automaton is circular if there exists a letter $a \in \Sigma$ that induces a circular permutation over the set of states of the automaton. Pin [8] proved that Černý conjecture holds true for circular automata with a prime number of states. This result was improved by Dubuc [9], who proved the conjecture for all circular automata.
2. **Aperiodic automata**: an automaton is aperiodic, if and only if, its transition semigroup is aperiodic. Trahtman [10] proved that Černý conjecture holds true for aperiodic automata.
3. **One-cluster automaton**: an automaton is a one-cluster automaton if there exits $a \in \Sigma$ such that the graph of a is connected. Steinberg [11] proved that Černý conjecture holds true for all the one-cluster automata for which the length of the one-cluster cycle is a prime number. An important application of synchronizing automata is related to code synchronization. Béal [12] proved that the fast synchronization of codes is related to the fast synchronization of one-cluster automata.
4. **Eulerian automata**: an automaton is Eulerian if the associated transition digraph is Eulerian. Kari proved that Eulerian automata are synchronizing, and he also proved that Černý conjecture holds true for this class of automata. Steinberg [13] proved how to deduce Kari's solution by a method for obtaining bounds on lengths of synchronizing words.
We consider that the above four classes constitute the most representative example of classes of automata for which it has been proved that Černý Conjecture holds true.
What is know about general automata ?
The first upper bound for general automata was proved by Černý who proved that the length of the shortest reset word not exceed the value $2^n - n - 1$. This result has been improved several times. Pirická and Rosenauerová proved in 1971 a better upper bound, they proved that the reset length is upperbounded by $\frac{1}{3}n^3 - \frac{3}{2}n^2 + \frac{25}{6} - 4$. Pin established a better bound in 1981, namely $(\frac{1}{2} - \frac{\pi}{36})n^3 + o(n^3)$. Then, in 1982, Pin [14] and Frankl [15] found a better bound $\frac{n^3-n}{6}$. Quite recently, in 2018, Szykula improved this bound achieving the best current upper bound: $\frac{114}{685}n^3 + O(n^2)$.
The Automata with reset word close to the Černý bound are very rare [16]. Then, it is natural to ask:
1. Are synchronizing automata frequent?
2. Does a synchronizing n-state automaton have a short synchronizing word with high probability?
Berlinkov [17] proved that the probability that a random n-state automaton is synchronizing is of the order $O(1 - n^{-\frac{1}{2}|\Sigma|})$. Nicaud [18] proved that when an automaton is chosen uniformly at random, the probability that it has a synchronizing word of length $O(n^{1+\epsilon})$ tends to 1 when n tends to infinity. Thus, it seems that most automata are synchronizing and can be synchronized with short reset words. Why are we interested in the existence of short synchronizing strings? Suppose one has to control a synchronizing automaton, it should include a reset word into his toolkit: if the system is taken out of control, it can use this reset word to drive the system towards a specific state that can be computed in advance. Notice that by doing so, the reset time will depend on the length of the reset word. Then, it is better if he chooses to compute a shortest synchronizing string.
Finally, it is important to remark that automata synchronization is not an isolated field, and that it has many application in other fields of applied computer science. One of the oldest and most surprising application is related to the setting of mechanical pieces (Natarajan [19]). There are applications related to coding (Berlinkov [12]), game theory (Maubert [20]) and matrix

mortality (Jungers [21]). Moreover, we have that automata synchronization is a special instance of the Rendezvous problem studied in distributed computing (Rajsbaum [22]).

Organization of the work, contributions and relations to previous work. This work is organized into four sections. In section 1 we introduce the main notions of subset synchronization and we state our main conjectures.In section 2 we focus on the synchronizing times of triples, and we begin the study of The $\frac{2}{3}$-*Černý* Conjecture. The latter conjecture states that given a n-state synchronizing automaton, the synchronizing time of its hardest to synchronize triples of states is bounded above by $\frac{2}{3}n^2 + o\left(n^2\right)$. In section 3 we study the case of Eulerian automata, we prove that a weak version of the conjecture holds true for almost all the Eulerian automata, and we finish this section with some concluding remarks that are related to planar Eulerian automata.

2. Subset synchronization

Let us recall the notion of deterministic finite state automaton.

Definition 1. *A DFA is a triple $\mathcal{M} = (Q_{\mathcal{M}}, \Sigma_{\mathcal{M}}, \delta_{\mathcal{M}})$ such that:*

- *$Q_{\mathcal{M}}$ is a finite set, the set of internal states of automaton $\mathcal{M}$.*
- *$\Sigma_{\mathcal{M}}$ is a finite alphabet, the input alphabet of $\mathcal{M}$.*
- *$\delta_{\mathcal{M}}$ is the transition function of $\mathcal{M}$, which is a function from $Q_{\mathcal{M}} \times \Sigma_{\mathcal{M}}$ to $Q_{\mathcal{M}}$.*

Definition 2. *Let $\mathcal{M} = (Q_{\mathcal{M}}, \Sigma_{\mathcal{M}}, \delta_{\mathcal{M}})$ be a DFA. We denote by $\Sigma_{\mathcal{M}}^*$ the set of finite strings over the alphabet $\Sigma_{\mathcal{M}}$. The function $\widehat{\delta_{\mathcal{M}}} : Q_{\mathcal{M}} \times \Sigma_{\mathcal{M}}^* \to Q_{\mathcal{M}}$, defined by the recursion:*

$$\widehat{\delta_{\mathcal{M}}}\left(q, w_1...w_n\right) = \delta_{\mathcal{M}}\left(\widehat{\delta_{\mathcal{M}}}\left(q, w_1...w_{n-1}\right), w_n\right),$$
$$\widehat{\delta_{\mathcal{M}}}\left(q, w_1\right) = \delta_{\mathcal{M}}(q, w_1).$$

Function $\widehat{\delta_{\mathcal{M}}}$ determines the state that is reached when automaton scans the string $w_1...w_n$, beginning in the state q.

Definition 3. *We say that an automaton $\mathcal{M}$ is synchronizing, if and only if, there exists a synchronizing string $w \in \Sigma_{\mathcal{M}}^*$, such that for all $p, q \in Q_{\mathcal{M}}$, the equality $\widehat{\delta_{\mathcal{M}}}\left(p, w\right) = \widehat{\delta_{\mathcal{M}}}\left(q, w\right)$ holds.*

of k agents is exploring an automaton $\mathcal{M}$. Suppose also that one wants to force all those agents to meet at a certain unspecified state. Then, if one can broadcast a single message, the same one for the k agents, he must choose to broadcast a string that synchronizes the locations (states) of those agents. We can consider two possible scenarios. In the first scenario one does not know the specific locations of the agents scattered through the territory, while in the second scenario he knows those k locations. The first scenario is equivalent to the classical Černý's scenario because the only possible solution is to synchronize all the states of $\mathcal{M}$. The second one corresponds to subset synchronization, it corresponds to synchronizing the k locations of the agents.

Definition 4. *Let $\mathcal{M}$ be a synchronizing automaton and let $q_1, q_2, ..., q_k \in Q_{\mathcal{M}}$.*

- *We use the symbol $st\left(\mathcal{M}, q_1, q_2, ..., q_k\right)$ to denote the length of a minimal synchronizing string for those k states.*
- *We use the symbol $st_k\left(\mathcal{M}\right)$ to denote the quantity*

  $$\max\left\{st\left(\mathcal{M}, q_1, q_2, ..., q_k\right) : q_1, q_2, ..., q_k \in Q_{\mathcal{M}}\right\},$$

  which is equal to the synchronizing time required by the hardest to synchronize k-tuple of states of automaton $\mathcal{M}$. We say that $st_k\left(\mathcal{M}\right)$ is the k-tuple rendezvous time of $\mathcal{M}$.

- *We use the symbol st_k to denote the function defined by*

$$st_k(n) = \max\{st_k(\mathcal{M}) : \mathcal{M} \textit{ is a } n\textit{-state synchronizing automaton}\},$$

  which we call the k-tuple rendezvous time function.

By an abuse of language we say that $st(\mathcal{M}, q_1, q_2, ..., q_k)$ is the *synchronizing time* of the tuple $\{q_1, q_2, ..., q_k\}$.
We study the sequence $\{st_k\}_{k \geq 2}$. We focus on the function $\mathcal{RT} : \mathbb{N} \to \mathbb{R}$ that is defined by

$$\mathcal{RT}(k) = \left\{ \begin{array}{c} 0, \text{ if } k = 0, 1 \\ \lim_{n\to\infty} \frac{st_k(n)}{n^2}, \text{ otherwise} \end{array} \right.$$

3. Eulerian automata

The references [23], [24] y [25] make a detailed study of the $\frac{2}{3}$-conjecture for the cases of circular automata and one cluster automata.In this section we discuss the remaining result, that is: we discuss some facts that are related to the 3-tuple rendezvous time of Eulerian automata.

Definition 5. *An automaton $\mathcal{M}$ is said to be Eulerian, if and only if, the transition digraph of $\mathcal{M}$ is Eulerian (parallel edges are not allowed).*

Kari proved that Černý Conjecture holds true for Eulerian automata [26], he proved that $(n-2)(n-1)+1$ is an upper bound on the synchronizing time of those automata. It is clear that Kari's upper bound is not strong enough as to imply that $\frac{2n^2}{3} + o(n)$ is an upper bound on the triple rendezvous time of Eulerian automata. Can we prove that the $\frac{2}{3}$-Černý Conjecture holds true for Eulerian automata?. Let $\mathcal{C}$ be a class of synchronizing automata. We know that $\mathcal{RT}_{\mathcal{C}}(3) \leq 1$. Suppose we prove that $\mathcal{RT}_{\mathcal{C}}(3) \leq 1 - \varepsilon$. If $\varepsilon \geq \frac{1}{3}$ we get that $\mathcal{RT}_{\mathcal{C}}(3) \leq \frac{2}{3}$ and we get that the $\frac{2}{3}$-Černý Conjecture holds true for the class $\mathcal{C}$. If $0 < \varepsilon < \frac{1}{3}$ we get a weaker but highly nontrivial result. In this section we prove a weak result concerning the 3-tuple rendezvous time of Eulerian automata.We prove that if p is prime, then it is *almost sure* that for all $\varepsilon > 0$ the quantity $(\frac{3}{4} + \varepsilon)p^2$ is an upper bound on the triple rendezvous time of p-state Eulerian automata.
It is known that the expected synchronizing time of n-state synchronizing automata is $O(n)$ [17]. The latter fact implies that Černý upper bound holds true with probability one. It also implies that the $\frac{2n^2}{3}$ upper bound on the 3-tuple rendezvous time holds with probability one. However, we have to notice that the set of Eulerian automata is a set of probability zero. Thus, the aforementioned probabilistic result does not have implications on the expected synchronizing time and the expected 3-tuple rendezvous time of Eulerian automata. If we want to prove probabilistic results for Eulerian automata we have to prove them from scratch.
We would like to prove (at least) that The $\frac{2}{3}$-Cerný Conjecture holds true for Eulerian automata with a high probability. We use some of the previous facts and ideas to get a weaker result. Our strategy reduces to show that given $\varepsilon > 0$, given p prime and given a p-state Eulerian automaton $\mathcal{M}$ the inequality $\alpha_{\mathcal{M}} \geq \frac{p}{4} - \varepsilon$ holds with a high probability.
We have to show as well that the probability of $\mathcal{G}$ being synchronizing goes to one when p goes to infinity.

Remark 1. *From now on, and for the ease of computations, we focus on the binary case.*

3.1. $\mathcal{RT}_{p\text{-}\mathcal{E}}(3) \leq \frac{3}{4}$ holds with probability one.

Let $\mathcal{G}$ be an Eulerian digraph with p nodes and suppose that for all v the equalities

$$\deg^+_{\mathcal{G}}(v) = \deg^-_{\mathcal{G}}(v) = 2$$

hold. We say in the later case that $\mathcal{G}$ is an *Eulerian frame*. A *road coloring* of $\mathcal{G}$ corresponds to assign a color a or b to each one of the edges in $\mathcal{G}$, the assignment must satisfy the following constraint: given a node v, its two outgoing edges are assigned different colors. Notice that if we choose a road coloring of $\mathcal{G}$, say c, we are simply choosing one of the Eulerian automata that can be constructed over the fixed topology determined by frame $\mathcal{G}$. We use the symbol $\mathcal{G}^c$ to denote the latter automaton.

We say that a road coloring c is synchronizing, if and only if, automaton $\mathcal{G}^c$ is synchronizing. Kari proved that given an Eulerian frame, there exists a road coloring of it that is synchronizing [26].

We suppose that the set of nodes of $\mathcal{G}$ is the set $\{1, ..., p\}$. Notice that it is easy to choose road colorings of $\mathcal{G}$ uniformly at random, it can be made in the following way:

- Let $v \in \{1, ..., p\}$ be a node, and let (v, i) and (v, j) be the two edges going out from v. Suppose that $i < j$ and set $v_1 = i$ and $v_2 = j$. Choose uniformly at random a bijection $f_v : \{1, 2\} \to \{a, b\}$. Given $i = 1, 2$, assign to edge (v, v_i) the color $f(i)$. Choose the functions $\{f_v : v \in \{1, ..., p\}\}$ in an independent way.

Let us fix a frame $\mathcal{G}$ with p nodes, and let $rc(\mathcal{G})$ be the set of road colorings of $\mathcal{G}$. We set:

(i) $\beta_{\mathcal{G}} = \Pr_{c \in_{\mathcal{U}} rc(\mathcal{G})} [\mathcal{G}^c \text{ is synchronizing}]$.

(ii) $\gamma_{\mathcal{G}}(\varepsilon) = \Pr_{c \in_{\mathcal{U}} rc(\mathcal{G})} \left[\alpha_{\mathcal{G}^c} \geq \frac{p}{4} - \varepsilon\right]$

We fix $\varepsilon > 0$, we want to prove that there exist two functions $r, s_\varepsilon : \mathbb{N} \to \mathbb{N}$ such that:

(i) $\beta_{\mathcal{G}} \geq 1 - r(p)$.

(ii) $\gamma_{\mathcal{G}}(\varepsilon) \geq 1 - s_\varepsilon(p)$.

(iii) $\lim_{p \to \infty} (r(p)) = \lim_{p \to \infty} (s_\varepsilon(p)) = 0$.

The analysis of $\gamma_{\mathcal{G}}(\varepsilon)$ is fairly easy. Let $w \in \{1, ..., p\}$ and let (i, w),(j, w) be the two edges going into w. Notice that the colors of those two edges were chosen in an independent way. Moreover, we have that

$$\Pr_c [c(i, w) = a] = \Pr_c [c(i, w) = b] = \tfrac{1}{2}.$$

Then, we have that

$$\Pr_c [c(i, w) = c(j, w) = a] = \Pr_c [c(i, w) = c(j, w) = b] = \tfrac{1}{4},$$

and it implies that the expected dimensions of $\mathbb{R}_a^{\mathcal{G}^c}$ and $\mathbb{R}_b^{\mathcal{G}^c}$ are both equal to $\frac{p}{4}$. It follows easily that there exists a function $s_\varepsilon(n)$ such that.

$$\left(\Pr_c \left[\alpha_{\mathcal{G}^c} \geq \tfrac{p}{4} - \varepsilon\right]\right) \geq 1 - s_\varepsilon(p) \text{ and } \lim_{p \to \infty} (s_\varepsilon(p)) = 0.$$

The remaining task is a little bit more demanding.

Let c be a road coloring of $\mathcal{G}$, we say that $A \subset Q$ is a *synchronizing subset of* $\mathcal{G}^c$, if and only if, there exists a string w such that the equality

$$|\delta_{\mathcal{G}^c}(A, w)| = 1$$

holds. We use the symbol $m_{\mathcal{G}^c}$ to denote the size of the largest synchronizing subsets of $\mathcal{G}^c$, that is:

$$m_{\mathcal{G}^c} = \max \{k : \text{there exists a synchronizing subset } A \text{ such that } |A| \geq k\}.$$

Notice that $\mathcal{G}^c$ is synchronizing, if and only if, the equality $m_{\mathcal{G}^c} = p$ holds. Thus, we want to prove that there exists a function $r : \mathbb{N} \to \mathbb{N}$ such that

$$\Pr_c [m_{\mathcal{G}^c} = p] \geq 1 - r(p) \text{ and } \lim_{n \to \infty} r(p) = 0.$$

Lemma 1. *For all prime p and for all Eulerian frame $\mathcal{G}$ of size p the equality*

$$\Pr_c [m_{\mathcal{G}^c} = n] = \Pr_c [m_{\mathcal{G}^c} \geq 2]$$

holds.

Proof. Kari's proved that for all road coloring c the number $m_{\mathcal{G}^c}$ divides p. Recall that p is prime, then we have that the equality $m_{\mathcal{G}^c} = p$ holds, if and only if, $m_{\mathcal{G}^c} \geq 2$. The equality

$$\Pr_c [m_{\mathcal{G}^c} = n] = \Pr_c [m_{\mathcal{G}^c} \geq 2]$$

follows easily from the latter fact. □

Lemma 2. $\Pr_c [m_{\mathcal{G}^c} \geq 2] \geq 1 - 2^{-p}$

Proof. Let s be a node, and let $(q, s), (r, s)$ be its two ingoing edges. If those two edges are assigned the same color, then $m_{\mathcal{G}^c} \geq 2$. Thus, we have that $\Pr_c [m_{\mathcal{G}^c} \geq 2]$ is lowerbounded by the probability that there exists a node s whose two ingoing edges are assigned the same color, we use the symbol $\alpha(p)$ to denote this probability. If we fix s, the probability of not assigning the same colors to its ingoing edges is $\frac{1}{2}$. It implies that $\alpha(p) \geq 1 - 2^{-p}$. Thus, we have that

$$\Pr_c [m_{\mathcal{G}^c} \geq 2] \geq \alpha(p) \geq 1 - 2^{-p},$$

and the lemma is proved. □

Given the above series of lemmata we can conclude that

Theorem 1. *Let Σ be an alphabet such that $|\Sigma| \geq 2$, let $\varepsilon > 0$ and let p be a prime number. Suppose that one chooses uniformly at random a p-state Eulerian automaton without parallel edges over the alphabet Σ. Let $P(p, \varepsilon)$ be the probability that the chosen automaton is synchronizing and that all its triples can be synchronized in time $\left(\frac{3}{4} + \varepsilon\right) p^2$. We have that $\lim_{p \to \infty} P(p, \varepsilon) = 1$.*

We can conclude, from the above theorem, that the inequality $\mathcal{RT}_{p\text{-}\mathcal{E}}(3) \leq \frac{3}{4}$ holds with probability one.

4. Conclusion

We are strongly convinced that the planarity constraint makes the problem become more tractable. One has to take into account that the class of planar digraphs is one of the most studied and best understood classes of digraphs (the class of planar digraphs is a very much more docile class than the class of strongly connected digraphs). We think that there are specific ways of proving Černý conjecture for planar automata (supposing it is true) that does not work for general automata. We would have liked to illustrate the latter point by proving the $\frac{2}{3}$-Černý conjecture for planar automata. Unfortunately, we could not prove this result. There is a way of illustrating our point in a very much weaker way: we conjecture that $\mathcal{RT}_{\mathcal{E}}(3) \leq \frac{2}{3}$. A possible attack to the latter conjecture is given by:

(i) Prove that the inequality $\mathcal{RT}_{P(\mathcal{E})}(3) \leq \frac{2}{3}$ implies the inequality $\mathcal{RT}_{\mathcal{E}}(3) \leq \frac{2}{3}$, where $P(\mathcal{E})$ is the class of planar Eulerian synchronizing automata.

(ii) Prove the inequality $\mathcal{RT}_{P(\mathcal{E})}(3) \leq \frac{2}{3}$.

We would like to observe that, in this special case, the planarity hypothesis makes the problem more tractable. The class of planar Eulerian digraphs (also called *planar Eulerian maps*) is a docile class. Thus it would be interesting to prove that $\mathcal{RT}_{P(\mathcal{E})}(3) \leq \frac{2}{3}$, using an argument strongly based on the planarity of the automata to be analyzed.

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