Fundamentals of Thermal-Fluid Sciences, 3rd Edition
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Chapter 3
NATURAL CONVECTION

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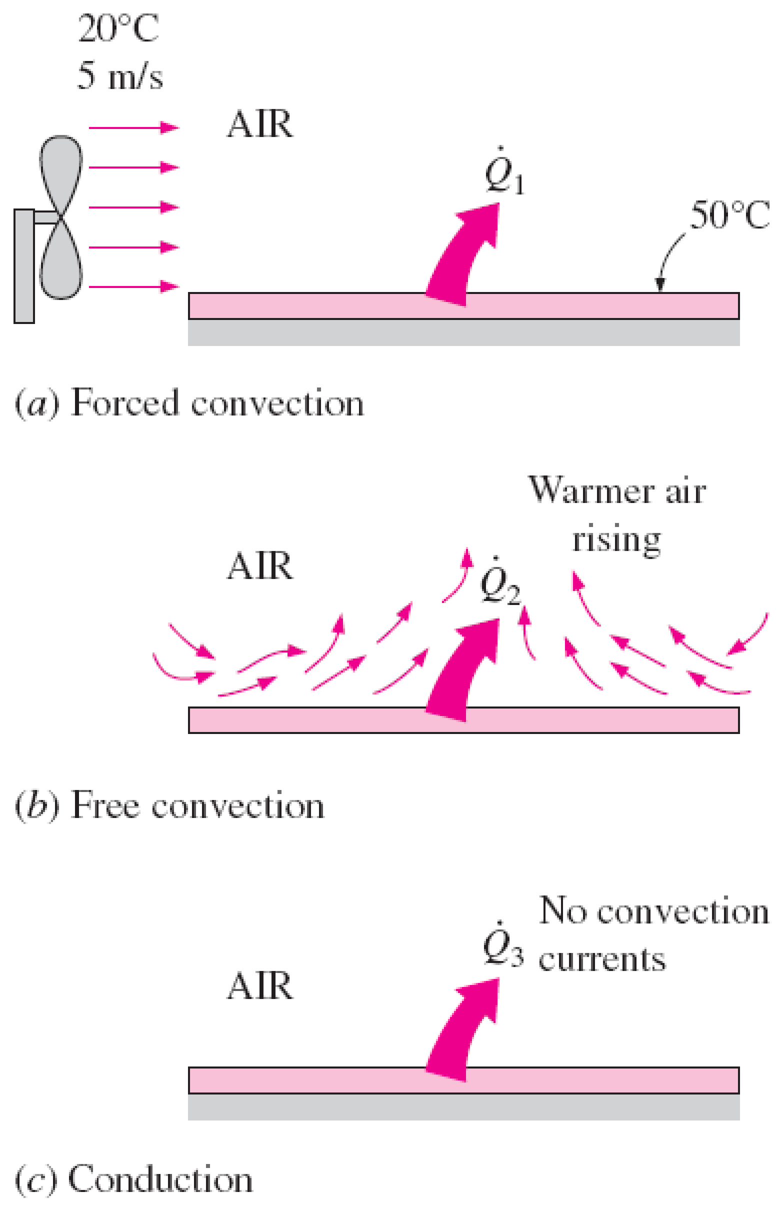
Objectives

- Understand the physical mechanism of natural convection
- Derive the governing equations of natural convection, and obtain the dimensionless Grashof number by nondimensionalizing them
- Evaluate the Nusselt number for natural convection associated with vertical, horizontal, and inclined plates as well as cylinders and spheres
- Examine natural convection from finned surfaces, and determine the optimum fin spacing
- Analyze natural convection inside enclosures such as double-pane windows.

PHYSICAL MECHANISM OF CONVECTION



Conduction and convection both require the presence of a material medium but convection requires fluid motion.

Convection involves fluid motion as well as heat conduction.

Heat transfer through a solid is always by conduction.

Heat transfer through a fluid is by convection in the presence of bulk fluid motion and by conduction in the absence of it.

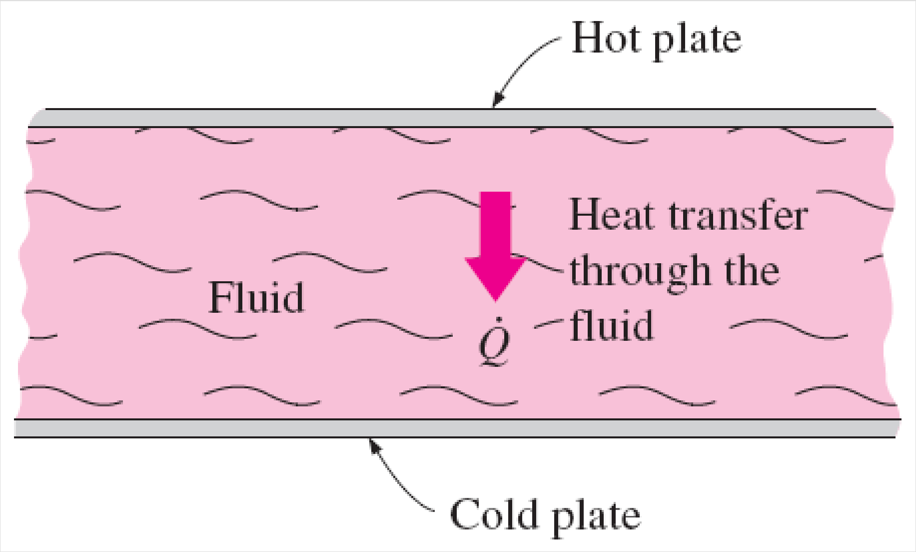
Therefore, conduction in a fluid can be viewed as the limiting case of convection, corresponding to the case of quiescent fluid.

Heat transfer from a hot surface to the surrounding fluid by convection and conduction.

The fluid motion enhances heat transfer, since it brings warmer and cooler chunks of fluid into contact, initiating higher rates of conduction at a greater number of sites in a fluid.

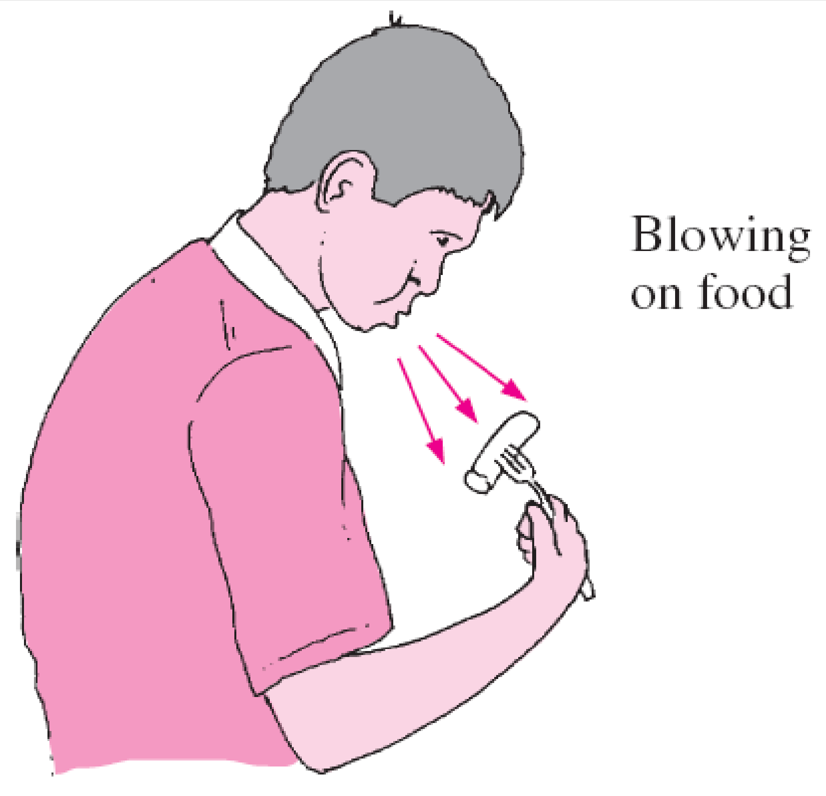
The rate of heat transfer through a fluid is much higher by convection than it is by conduction.

In fact, the higher the fluid velocity, the higher the rate of heat transfer.



Heat transfer through a fluid sandwiched between two parallel plates.

Convection in daily life



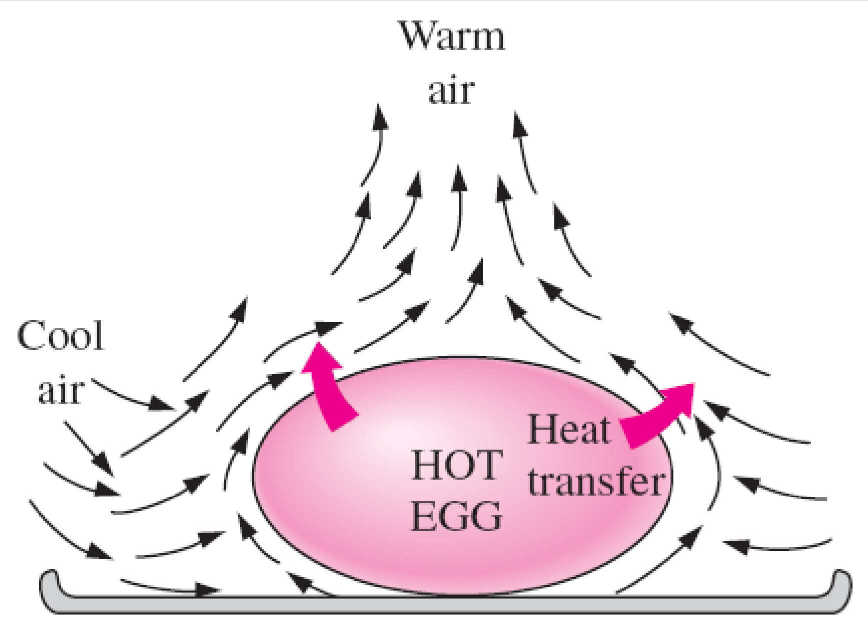
We resort to forced convection whenever we need to increase the rate of heat transfer.

- We turn on the fan on hot summer days to help our body cool more effectively. The higher the fan speed, the better we feel.
- We *stir* our soup and *blow* on a hot slice of pizza to make them cool faster.
- The air on windy winter days feels much colder than it actually is.
- The simplest solution to heating problems in electronics packaging is to use a large enough fan.

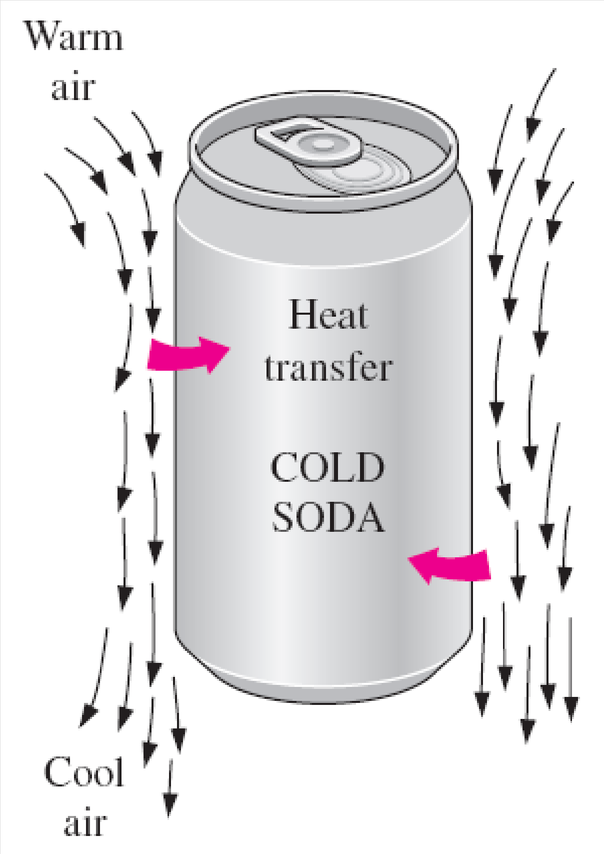
Many familiar heat transfer applications involve natural convection as the primary mechanism of heat transfer. Examples?

Natural convection in gases is usually accompanied by radiation of comparable magnitude except for low-emissivity surfaces.

The motion that results from the continual replacement of the heated air in the vicinity of the egg by the cooler air nearby is called a natural convection current, and the heat transfer that is enhanced as a result of this current is called natural convection heat transfer.



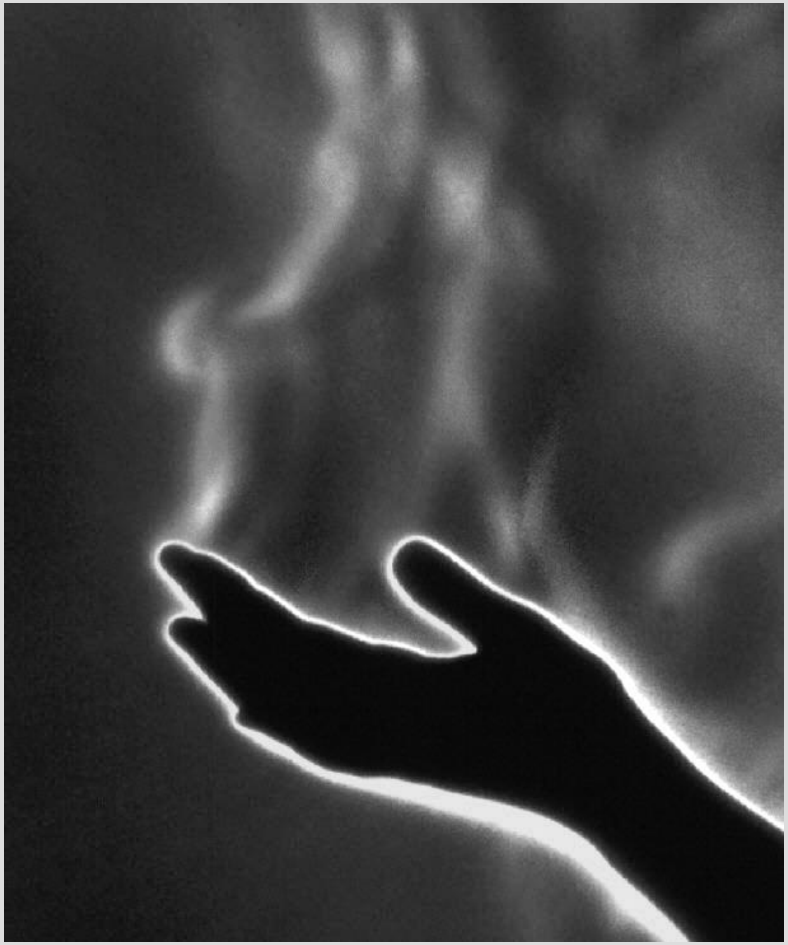
The cooling of a boiled egg in a cooler environment by natural convection.



The warming up of a cold drink in a warmer environment by natural convection.

The density of a fluid, in general, depends more strongly on temperature than it does on pressure, and the variation of density with temperature is resposible for numerous natural phenomena such a winds, currents in oceans, rise of plumes in chimneys, the operation of hot-air balloons, heat transfer by natural convection.

To quantify these effects, we need a property that represents the variation of the density of a fluid with temperature at constant pressure.

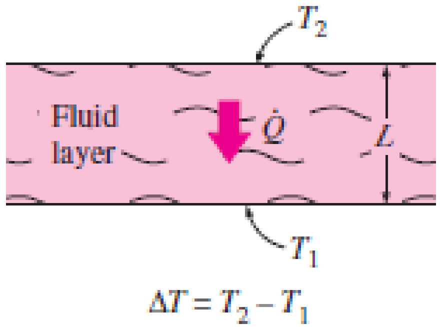


Natural convection over a woman’s hand.

Nusselt number

In convection studies, it is common practice to nondimensionalize the governing equations and combine the variables, which group together into *dimensionless numbers* in order to reduce the number of total variables. It is also common practice to nondimensionalize the heat transfer coefficient h with the Nusselt number, defined as

$$\mathrm{Nu} = \frac{hL_c}{k}$$



Heat transfer through a fluid layer of thickness L and temperature difference ΔT.

$$\dot{q}_{\text{conv}} = h\Delta T$$

and

$$\dot{q}_{\text{cond}} = k\frac{\Delta T}{L}$$

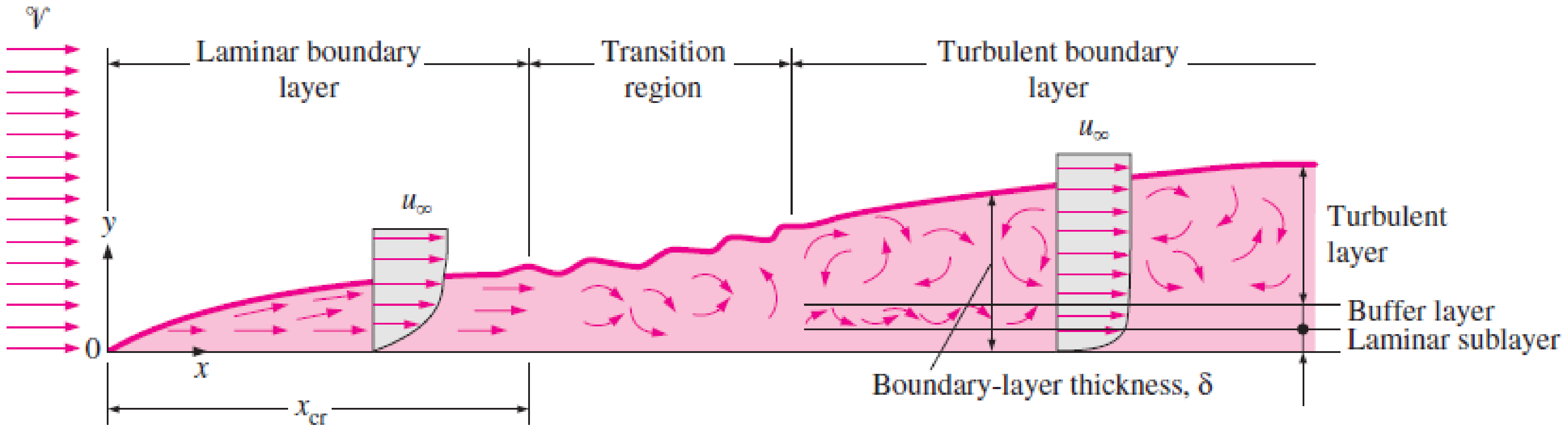
Taking their ratio gives

$$\frac{\dot{q}_{\text{conv}}}{\dot{q}_{\text{cond}}} = \frac{h\Delta T}{k\Delta T/L} = \frac{hL}{k} = \mathrm{Nu}$$

VELOCITY BOUNDARY LAYER

The region of the flow above the plate bounded by δ in which the effects of the viscous shearing forces caused by fluid viscosity are felt is called the **velocity boundary layer.** The *boundary layer thickness*, δ, is typically defined as the distance y from the surface at which $u = 0.99u_\infty$.

The hypothetical line of $u = 0.99u_\infty$ divides the flow over a plate into two regions: the **boundary layer region,** in which the viscous effects and the velocity changes are significant, and the **inviscid flow region,** in which the frictional effects are negligible and the velocity remains essentially constant.



The development of the boundary layer for flow over a flat plate, and the different flow regimes.

SURFACE SHEAR STRESS

Consider the flow of a fluid over the surface of a plate. The fluid layer in contact with the surface will try to drag the plate along via friction, exerting a *friction force* on it. Likewise, a faster fluid layer will try to drag the adjacent slower layer and exert a friction force because of the friction between the two layers. Friction force per unit area is called **shear stress,** and is denoted by τ. Experimental studies indicate that the shear stress for most fluids is proportional to the *velocity gradient*, and the shear stress at the wall surface is as

$$\tau_s = \mu \left. \frac{\partial u}{\partial y} \right|_{y=0} \qquad (\text{N/m}^2)$$

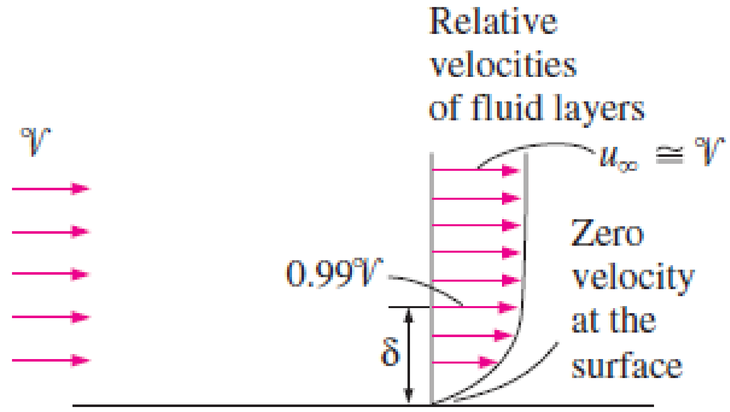


FIGURE 6–11
The development of a boundary layer on a surface is due to the no-slip condition.

SURFACE SHEAR STRESS

The determination of the surface shear stress τ_s from Eq. 6-9 is not practical since it requires a knowledge of the flow velocity profile. A more practical approach in external flow is to relate τ_s to the upstream velocity $\mathcal{V}$ as

$$\tau_s = C_f \frac{\rho \mathcal{V}^2}{2} \qquad (\text{N/m}^2)$$

where C_f is the dimensionless **friction coefficient,** whose value in most cases is determined experimentally, and ρ is the density of the fluid. Note that the friction coefficient, in general, will vary with location along the surface. Once the average friction coefficient over a given surface is available, the friction force over the entire surface is determined from

$$F_f = C_f A_s \frac{\rho \mathcal{V}^2}{2} \qquad (\text{N})$$

where A_s is the surface area.

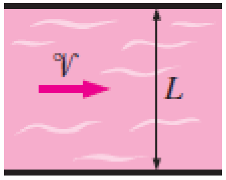
The friction coefficient is an important parameter in heat transfer studies since it is directly related to the heat transfer coefficient and the power requirements of the pump or fan.

Reynolds number

The transition from laminar to turbulent flow depends on the *surface geometry, surface roughness, free-stream velocity, surface temperature*, and *type of fluid*, among other things. After exhaustive experiments in the 1880s, Osborn Reynolds discovered that the flow regime depends mainly on the ratio of the *inertia forces* to *viscous forces* in the fluid. This ratio is called the **Reynolds number,** which is a *dimensionless* quantity, and is expressed for external flow as (Fig. 6–16)

$$\text{Re} = \frac{\text{Inertia forces}}{\text{Viscous}} = \frac{\mathcal{V} L_c}{\nu} = \frac{\rho \mathcal{V} L_c}{\mu}$$

where $\mathcal{V}$ is the upstream velocity (equivalent to the free-stream velocity u_∞ for a flat plate), L_c is the characteristic length of the geometry, and $\nu = \mu/\rho$ is the kinematic viscosity of the fluid. For a flat plate, the characteristic length is the distance x from the leading edge.



$$\text{Re} = \frac{\text{Inertia forces}}{\text{Viscous forces}} = \frac{\rho \mathcal{V}^2/L}{\mu \mathcal{V}/L^2} = \frac{\rho \mathcal{V} L}{\mu} = \frac{\mathcal{V} L}{\nu}$$

THERMAL BOUNDARY LAYER

The flow region over the surface in which the temperature variation in the direction normal to the surface is significant is the **thermal boundary layer.** The *thickness* of the thermal boundary layer δ_t at any location along the surface is defined as *the distance from the surface at which the temperature difference* $T - T_s$ *equals* $0.99(T_\infty - T_s)$. Note that for the special case of $T_s = 0$, we have $T = 0.99T_\infty$ at the outer edge of the thermal boundary layer, which is analogous to $u = 0.99u_\infty$ for the velocity boundary layer.

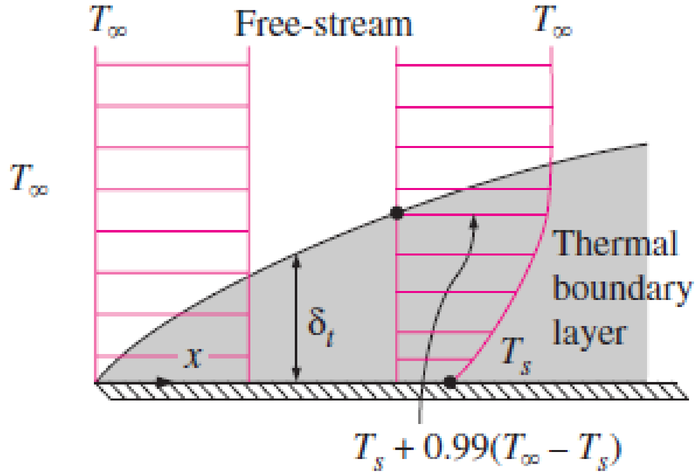


FIGURE 6–13
Thermal boundary layer on a flat plate (the fluid is hotter than the plate surface).

THERMAL BOUNDARY LAYER

Prandtl Number

The relative thickness of the velocity and the thermal boundary layers is best described by the *dimensionless* parameter **Prandtl number,** defined as

$$\Pr = \frac{\text{Molecular diffusivity of momentum}}{\text{Molecular diffusivity of heat}} = \frac{\nu}{\alpha} = \frac{\mu C_p}{k}$$

Typical ranges of Prandtl numbers for common fluids

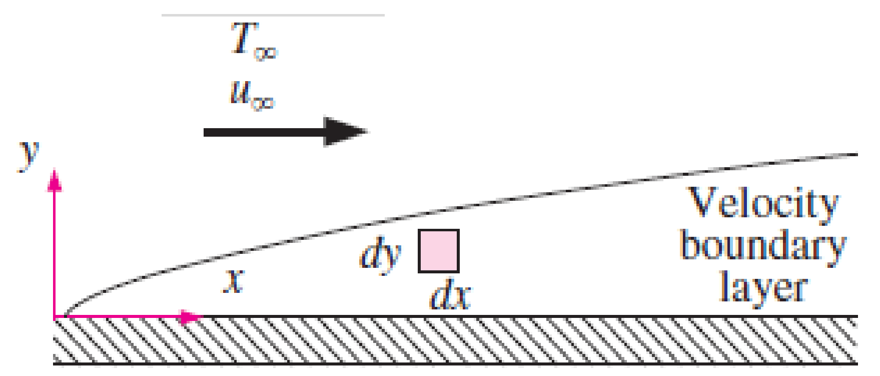
Fluid	Pr
Liquid metals	0.004–0.030
Gases	0.7–1.0
Water	1.7–13.7
Light organic fluids	5–50
Oils	50–100,000
Glycerin	2000–100,000

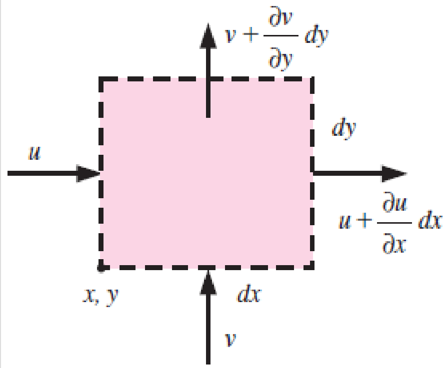
DERIVATION OF DIFFERENTIAL CONVECTION EQUATIONS*

Conservation of Mass Equation

The conservation of mass principle is simply a statement that mass cannot be created or destroyed, and all the mass must be accounted for during an analysis. In steady flow, the amount of mass within the control volume remains constant, and thus the conservation of mass can be expressed as

$$\begin{pmatrix}\text{Rate of mass flow}\\ \text{into the control volume}\end{pmatrix} = \begin{pmatrix}\text{Rate of mass flow}\\ \text{out of the control volume}\end{pmatrix}$$

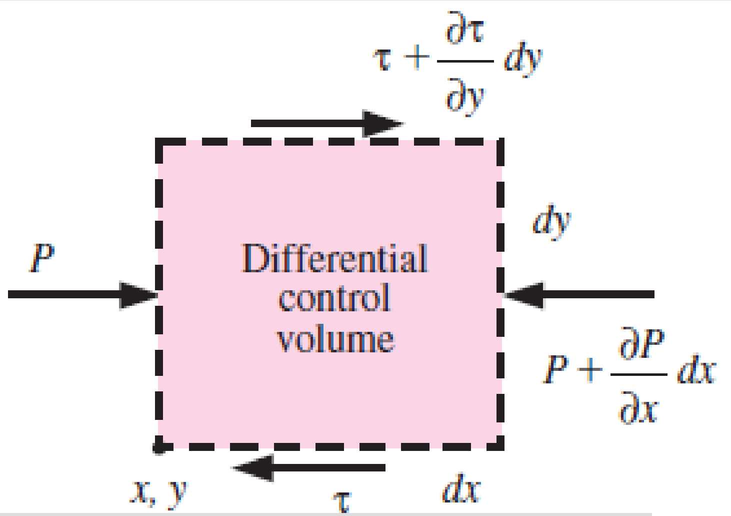




$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

DERIVATION OF DIFFERENTIAL CONVECTION EQUATIONS*

Conservation of Momentum Equations



Forces acting on a differential volume element in the natural convection boundary layer over a vertical flat plate.

$$\delta m \cdot a_x = F_{\text{surface}, x} + F_{\text{body}, x}$$

$$\delta m = \rho(dx \cdot dy \cdot 1)$$

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$$

$$F_{\text{surface}, x} = \left(\frac{\partial \tau}{\partial y} dy\right)(dx \cdot 1) - \left(\frac{\partial P}{\partial x} dx\right)(dy \cdot 1) = \left(\frac{\partial \tau}{\partial y} - \frac{\partial P}{\partial x}\right)(dx \cdot dy \cdot 1)$$
$$= \left(\mu\frac{\partial^2 u}{\partial y^2} - \frac{\partial P}{\partial x}\right)(dx \cdot dy \cdot 1)$$

since $\tau = \mu(\partial u/\partial y)$. dividing by $dx \cdot dy \cdot 1$ gives

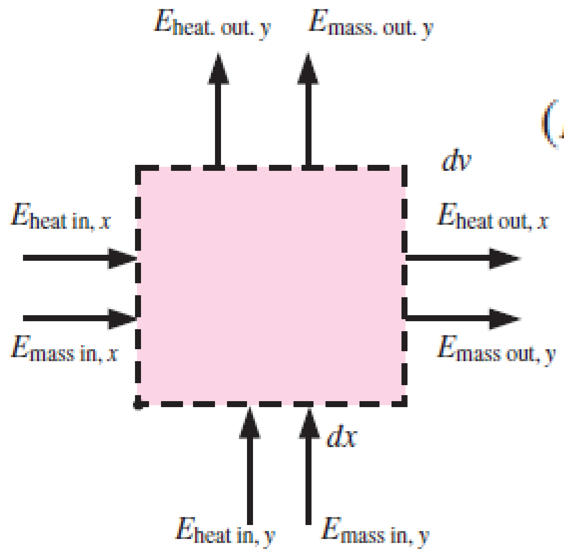
$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \mu\frac{\partial^2 u}{\partial y^2} - \frac{\partial P}{\partial x}$$

DERIVATION OF DIFFERENTIAL CONVECTION EQUATIONS*

Conservation of Energy Equation

The energy balance for any system undergoing any process is expressed as $E_{\text{in}} - E_{\text{out}} = \Delta E_{\text{system}}$, which states that the change in the energy content of a system during a process is equal to the difference between the energy input and the energy output. During a *steady-flow process,* the total energy content of a control volume remains constant (and thus $\Delta E_{\text{system}} = 0$), and the amount of energy entering a control volume in all forms must be equal to the amount of energy leaving it. Then the rate form of the general energy equation reduces for a steady-flow process to $\dot{E}_{\text{in}} - \dot{E}_{\text{out}} = 0$.

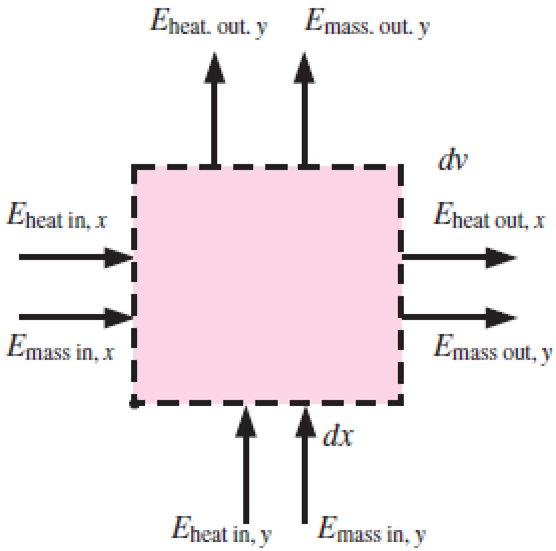
Noting that energy can be transferred by heat, work, and mass only, the energy balance for a steady-flow control volume can be written explicitly as



$$(\dot{E}_{\text{in}} - \dot{E}_{\text{out}})_{\text{by heat}} + (\dot{E}_{\text{in}} - \dot{E}_{\text{out}})_{\text{by work}} + (\dot{E}_{\text{in}} - \dot{E}_{\text{out}})_{\text{by mass}} = 0$$

DERIVATION OF DIFFERENTIAL CONVECTION EQUATIONS*

Conservation of Energy Equation



$$(\dot{E}_{\text{in}} - \dot{E}_{\text{out}})_{\text{by mass},\, x} = (\dot{m}e_{\text{stream}})_x - \left[(\dot{m}e_{\text{stream}})_x + \frac{\partial(\dot{m}e_{\text{stream}})_x}{\partial x}dx\right]$$

$$= -\frac{\partial[\rho u(dy\cdot 1)C_pT]}{\partial x}dx = -\rho C_p\left(u\frac{\partial T}{\partial x} + T\frac{\partial u}{\partial x}\right)dx\,dy$$

$$(\dot{E}_{\text{in}} - \dot{E}_{\text{out}})_{\text{by mass}} = -\rho C_p\left(u\frac{\partial T}{\partial x} + T\frac{\partial u}{\partial x}\right)dx\,dy - \rho C_p\left(v\frac{\partial T}{\partial y} + T\frac{\partial v}{\partial y}\right)dx\,dy$$

$$= -\rho C_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right)dx\,dy$$

$$(\dot{E}_{\text{in}} - \dot{E}_{\text{out}})_{\text{by heat},\, x} = \dot{Q}_x - \left(\dot{Q}_x + \frac{\partial \dot{Q}_x}{\partial x}dx\right)$$

$$= -\frac{\partial}{\partial x}\left(-k(dy\cdot 1)\frac{\partial T}{\partial x}\right)dx = k\frac{\partial^2 T}{\partial x^2}dx\,dy$$

$$(\dot{E}_{\text{in}} - \dot{E}_{\text{out}})_{\text{by heat}} = k\frac{\partial^2 T}{\partial x^2}dx\,dy + k\frac{\partial^2 T}{\partial y^2}dx\,dy = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)dx\,dy$$

DERIVATION OF DIFFERENTIAL CONVECTION EQUATIONS*

Conservation of Energy Equation

Then the energy equation for the steady two-dimensional flow of a fluid with constant properties and negligible shear stresses is obtained by substituting Eqs. 6-32 and 6-34 into 6-30 to be

$$\rho C_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

which states that *the net energy convected by the fluid out of the control volume is equal to the net energy transferred into the control volume by heat conduction.*

When the viscous shear stresses are not negligible, their effect is accounted for by expressing the energy equation as

$$\rho C_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \mu\Phi$$

where the *viscous dissipation function* Φ is obtained after a lengthy analysis (see an advanced book such as the one by *Schlichting* (Ref. 9) for details) to be

$$\Phi = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2$$

DERIVATION OF DIFFERENTIAL CONVECTION EQUATIONS*

Conservation of Energy Equation

Viscous dissipation may play a dominant role in high-speed flows, especially when the viscosity of the fluid is high (like the flow of oil in journal bearings). This manifests itself as a significant rise in fluid temperature due to the conversion of the kinetic energy of the fluid to thermal energy. Viscous dissipation is also significant for high-speed flights of aircraft.

For the special case of a stationary fluid, $u = v = 0$ and the energy equation reduces, as expected, to the steady two-dimensional heat conduction equation,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

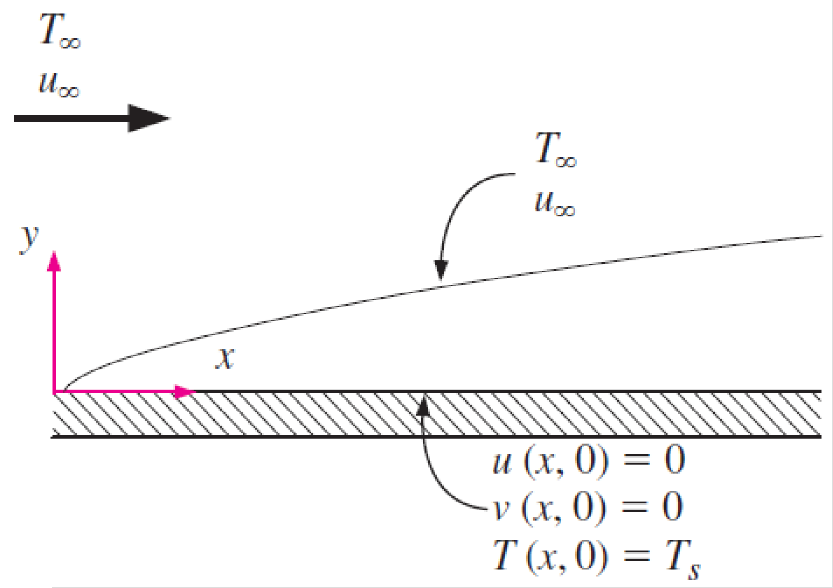
SOLUTION OF CONVECTION EQUATION FOR A FLAT PLATE

When viscous dissipation is negligible, the continuity, momentum, and energy equations reduce for steady, incompressible, laminar flow of a fluid with constant properties over a flat plate to

Continuity: $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Momentum: $$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

Energy: $$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$$



with the boundary conditions

At $x = 0$: $u(0, y) = u_\infty$, $T(0, y) = T_\infty$

At $y = 0$: $u(x, 0) = 0$, $v(x, 0) = 0$, $T(x, 0) = T_s$

As $y \rightarrow \infty$: $u(x, \infty) = u_\infty$, $T(x, \infty) = T_\infty$

SOLUTION OF CONVECTION EQUATION FOR A FLAT PLATE

The continuity and momentum equations were first solved in 1908 by the German engineer H. Blasius, a student of L. Prandtl. This was done by transforming the two partial differential equations into a single ordinary differential equation by introducing a new independent variable, called the **similarity variable.** The finding of such a variable, assuming it exists, is more of an art than science, and it requires to have a good insight of the problem.

although both δ and u at a given y vary with x, the velocity u at a fixed y/δ remains constant. Blasius was also aware from the work of Stokes that δ is proportional to $\sqrt{\nu x/u_\infty}$, and thus he defined a *dimensionless similarity variable* as

$$\eta = y\sqrt{\frac{u_\infty}{\nu x}}$$

and thus u/u_∞ = function(η). He then introduced a *stream function* $\psi(x, y)$

SOLUTION OF CONVECTION EQUATION FOR A FLAT PLATE

$$\eta = y\sqrt{\frac{u_\infty}{\nu x}}$$

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}$$

$$f(\eta) = \frac{\psi}{u_\infty \sqrt{\nu x / u_\infty}}$$

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y} = u_\infty \sqrt{\frac{\nu x}{u_\infty}}\frac{df}{d\eta}\sqrt{\frac{u_\infty}{\nu x}} = u_\infty \frac{df}{d\eta}$$

$$v = -\frac{\partial \psi}{\partial x} = -u_\infty \sqrt{\frac{\nu x}{u_\infty}}\frac{\partial f}{\partial x} - \frac{u_\infty}{2}\sqrt{\frac{\nu}{u_\infty x}}\, f = \frac{1}{2}\sqrt{\frac{u_\infty \nu}{x}}\left(\eta \frac{df}{d\eta} - f\right)$$

$$\frac{\partial u}{\partial x} = -\frac{u_\infty}{2x}\eta \frac{d^2 f}{d\eta^2}, \qquad \frac{\partial u}{\partial y} = u_\infty \sqrt{\frac{u_\infty}{\nu x}}\frac{d^2 f}{d\eta^2}, \qquad \frac{\partial^2 u}{\partial y^2} = \frac{u_\infty^2}{\nu x}\frac{d^3 f}{d\eta^3}$$

$$2\frac{d^3 f}{d\eta^3} + f\frac{d^2 f}{d\eta^2} = 0$$

$$f(0) = 0, \qquad \left.\frac{df}{d\eta}\right|_{\eta=0} = 0, \qquad \text{and} \qquad \left.\frac{df}{d\eta}\right|_{\eta=\infty} = 1$$

Similarity function f and its derivatives for laminar boundary layer along a flat plate.

η	f	$\frac{df}{d\eta} = \frac{u}{u_\infty}$	$\frac{d^2 f}{d\eta^2}$
0	0	0	0.332
0.5	0.042	0.166	0.331
1.0	0.166	0.330	0.323
1.5	0.370	0.487	0.303
2.0	0.650	0.630	0.267
2.5	0.996	0.751	0.217
3.0	1.397	0.846	0.161
3.5	1.838	0.913	0.108
4.0	2.306	0.956	0.064
4.5	2.790	0.980	0.034
5.0	3.283	0.992	0.016
5.5	3.781	0.997	0.007
6.0	4.280	0.999	0.002
∞	∞	1	0

SOLUTION OF CONVECTION EQUATION FOR A FLAT PLATE

the velocity boundary layer thickness becomes

$$\delta = \frac{5.0}{\sqrt{u_\infty / \nu x}} = \frac{5.0x}{\sqrt{\mathrm{Re}_x}}$$

since $\mathrm{Re}_x = u_\infty x/\nu$, where x is the distance from the leading edge of the plate. Note that the boundary layer thickness increases with increasing kinematic viscosity ν and with increasing distance from the leading edge x, but it decreases with increasing free-stream velocity u_∞. Therefore, a large free-stream velocity will suppress the boundary layer and cause it to be thinner.

The shear stress on the wall can be determined from

$$\tau_w = 0.332 u_\infty \sqrt{\frac{\rho \mu u_\infty}{x}} = \frac{0.332 \rho u_\infty^2}{\sqrt{\mathrm{Re}_x}}$$

Then the local skin friction coefficient becomes

$$C_{f,x} = \frac{\tau_w}{\rho \mathcal{V}^2/2} = \frac{\tau_w}{\rho u_\infty^2/2} = 0.664\ \mathrm{Re}_x^{-1/2}$$

Note that unlike the boundary layer thickness, wall shear stress and the skin friction coefficient decrease along the plate as $x^{-1/2}$.

SOLUTION OF CONVECTION EQUATION FOR A FLAT PLATE

The Energy Equation

Knowing the velocity profile, we are now ready to solve the energy equation for temperature distribution for the case of constant wall temperature T_s. First we introduce the dimensionless temperature θ as

$$\theta(x, y) = \frac{T(x, y) - T_s}{T_\infty - T_s}$$

Noting that both T_s and T_∞ are constant, substitution into the energy equation gives

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \alpha\frac{\partial^2 \theta}{\partial y^2}$$

$$2\frac{d^2\theta}{d\eta^2} + \Pr f\frac{d\theta}{d\eta} = 0$$

CONVECTION FOR A FLAT PLATE

Then the local convection coefficient and Nusselt number become

$$h_x = \frac{q_s}{T_s - T_\infty} = \frac{-k(\partial T/\partial y)|_{y=0}}{T_s - T_\infty} = 0.332\,\mathrm{Pr}^{1/3} k \sqrt{\frac{u_\infty}{\nu x}}$$

$$\bar{h} = \frac{1}{L}\int_0^L h\,dx$$

and

$$\mathrm{Nu}_x = \frac{h_x x}{k} = 0.332\,\mathrm{Pr}^{1/3}\mathrm{Re}_x^{1/2} \qquad \mathrm{Pr} > 0.6$$

Then the thermal boundary layer thickness becomes

$$\delta_t = \frac{\delta}{\mathrm{Pr}^{1/3}} = \frac{5.0x}{\mathrm{Pr}^{1/3}\sqrt{\mathrm{Re}_x}}$$

Note that these relations are valid only for laminar flow over an isothermal flat plate. Also, the effect of variable properties can be accounted for by evaluating all such properties at the film temperature defined as $T_f = (T_s + T_\infty)/2$.

CONVECTION FOR A FLAT PLATE

NONDIMENSIONALIZED CONVECTION EQUATIONS AND SIMILARITY

$$x^* = \frac{x}{L}, \quad y^* = \frac{y}{L}, \quad u^* = \frac{u}{\mathcal{V}}, \quad v^* = \frac{v}{\mathcal{V}}, \quad P^* = \frac{P}{\rho \mathcal{V}^2}, \quad \text{and} \quad T^* = \frac{T - T_s}{T_\infty - T_s}$$

Continuity:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

Momentum:

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \frac{1}{\mathrm{Re}_L} \frac{\partial^2 u^*}{\partial y^{*2}} - \frac{dP^*}{dx^*}$$

Energy:

$$u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{1}{\mathrm{Re}_L \mathrm{Pr}} \frac{\partial^2 T^*}{\partial y^{*2}}$$

$$u^*(0, y^*) = 1, \quad u^*(x^*, 0) = 0, \quad u^*(x^*, \infty) = 1, \quad v^*(x^*, 0) = 0,$$
$$T^*(0, y^*) = 1, \quad T^*(x^*, 0) = 0, \quad T^*(x^*, \infty) = 1$$

CONVECTION FOR A FLAT PLATE

ANALOGIES BETWEEN MOMENTUM AND HEAT TRANSFER

Reconsider the nondimensionalized momentum and energy equations for steady, incompressible, laminar flow of a fluid with constant properties and negligible viscous dissipation (Eqs. 6-65 and 6-66). When Pr = 1 (which is approximately the case for gases) and $\partial P^*/\partial x^* = 0$ (which is the case when, $u = u_\infty = \mathcal{V} = constant$ in the free stream, as in flow over a flat plate), these equations simplify to

Momentum:
$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \frac{1}{\mathrm{Re}_L} \frac{\partial^2 u^*}{\partial y^{*2}}$$

Energy:
$$u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{1}{\mathrm{Re}_L} \frac{\partial^2 T^*}{\partial y^{*2}}$$

CONVECTION FOR A FLAT PLATE

ANALOGIES BETWEEN MOMENTUM AND HEAT TRANSFER

$$C_{f,x}\frac{\mathrm{Re}_L}{2} = \mathrm{Nu}_x \qquad (\mathrm{Pr} = 1)$$

which is known as the **Reynolds analogy** (Fig. 6–32). This is an important analogy since it allows us to determine the heat transfer coefficient for fluids with Pr ≈ 1 from a knowledge of friction coefficient which is easier to measure. Reynolds analogy is also expressed alternately as

$$\frac{C_{f,x}}{2} = \mathrm{St}_x \qquad (\mathrm{Pr} = 1)$$

where

$$\mathrm{St} = \frac{h}{\rho C_p \mathcal{V}} = \frac{\mathrm{Nu}}{\mathrm{Re}_L \mathrm{Pr}}$$

is the **Stanton number,** which is also a dimensionless heat transfer coefficient.

CONVECTION FOR A FLAT PLATE

ANALOGIES BETWEEN MOMENTUM AND HEAT TRANSFER

Reynolds analogy is of limited use because of the restrictions Pr = 1 and $\partial P^*/\partial x^* = 0$ on it, and it is desirable to have an analogy that is applicable over a wide range of Pr. This is done by adding a Prandtl number correction. The friction coefficient and Nusselt number for a flat plate are determined in Section 6-8 to be

$$C_{f,x} = 0.664\,\mathrm{Re}_x^{-1/2} \qquad \text{and} \qquad \mathrm{Nu}_x = 0.332\,\mathrm{Pr}^{1/3}\,\mathrm{Re}_x^{1/2}$$

Taking their ratio and rearranging give the desired relation, known as the **modified Reynolds analogy** or **Chilton–Colburn analogy,**

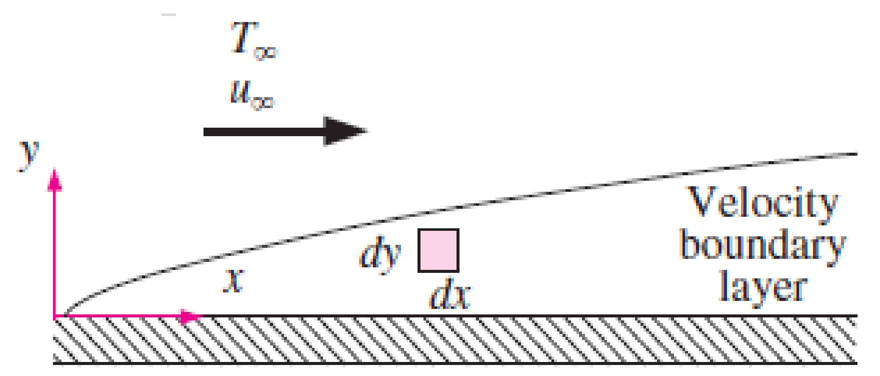
$$C_{f,x}\frac{Re_L}{2} = \mathrm{Nu}_x\,\mathrm{Pr}^{-1/3} \qquad \text{or} \qquad \frac{C_{f,x}}{2} = \frac{h_x}{\rho C_p \mathcal{V}}\mathrm{Pr}^{2/3} \equiv j_H$$

for $0.6 < \mathrm{Pr} < 60$. Here j_H is called the *Colburn j-factor.* Although this relation is developed using relations for laminar flow over a flat plate (for which $\partial P^*/\partial x^* = 0$), experimental studies show that it is also applicable approximately for turbulent flow over a surface, even in the presence of pressure gradients.

Quiz

1. Using the differential element shown in the figure, prove that in the hydroyinamic boundary layer the continuity equation is:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$



2. Define the Prandtl number and say what is its principal application in heat transfer problems.

Example

The flow of oil in a journal bearing can be approximated as parallel flow between two large plates with one plate moving and the other stationary. Such flows are known as Couette flow.

Consider two large isothermal plates separated by 2-mm-thick oil film. The upper plates moves at a constant velocity of 12 m/s, while the lower plate is stationary. Both plates are maintained at 20°C. (*a*) Obtain relations for the velocity and temperature distributions in the oil. (*b*) Determine the maximum temperature in the oil and the heat flux from the oil to each plate (Fig. 6–25).

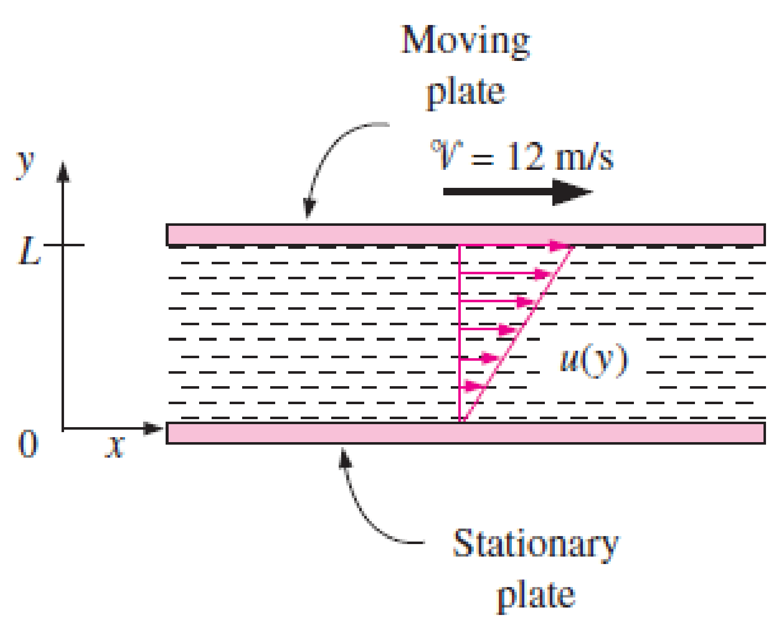
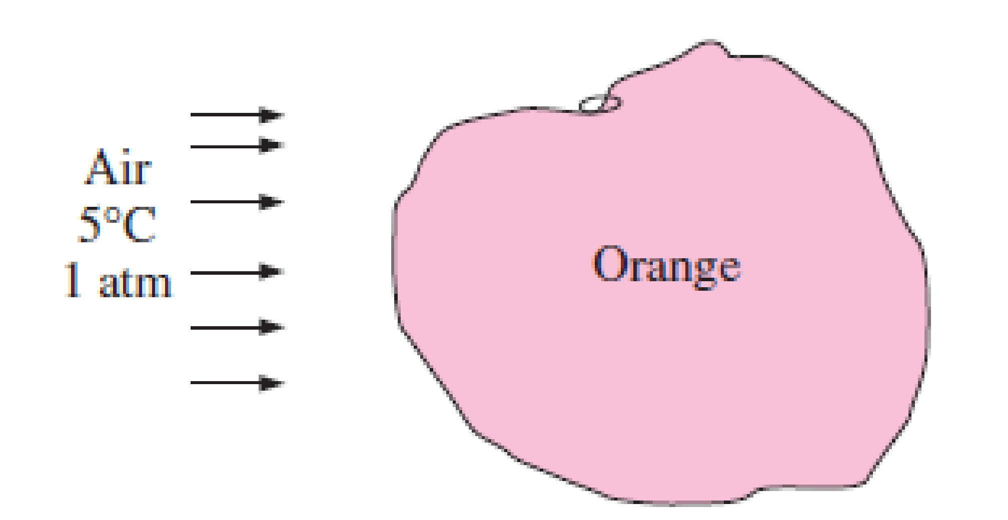


FIGURE 6–25

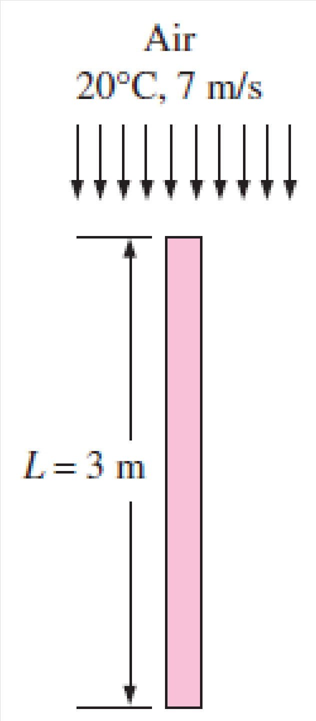
Exercise

6–11 During air cooling of oranges, grapefruit, and tangelos, the heat transfer coefficient for combined convection, radiation, and evaporation for air velocities of $0.11 < \mathcal{V} < 0.33$ m/s is determined experimentally and is expressed as $h = 5.05\ k_{air}\mathrm{Re}^{1/3}/D$, where the diameter D is the characteristic length. Oranges are cooled by refrigerated air at 5°C and 1 atm at a velocity of 0.5 m/s. Determine (*a*) the initial rate of heat transfer from a 7-cm-diameter orange initially at 15°C with a thermal conductivity of 0.50 W/m · °C, (*b*) the value of the initial temperature gradient inside the orange at the surface, and (*c*) the value of the Nusselt number.



Exercise

A 2-m × 3-m flat plate is suspended in a room, and is subjected to air flow parallel to its surfaces along its 3-m-long side. The free stream temperature and velocity of air are 20°C and 7 m/s. The total drag force acting on the plate is measured to be 0.86 N. Determine the average convection heat transfer coefficient for the plate (Fig. 6–33).

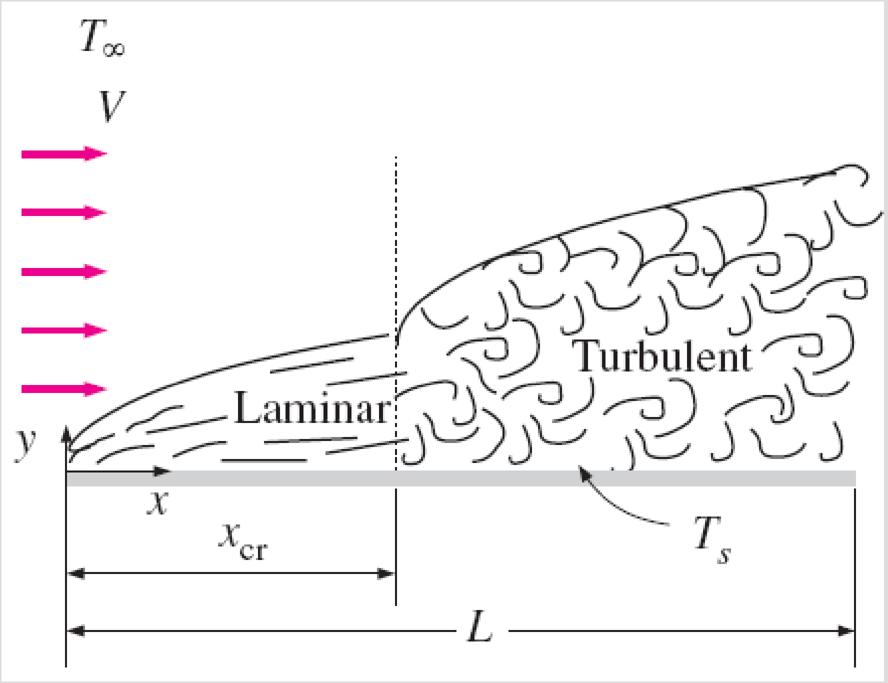


FORCED CONVECTION

PARALLEL FLOW OVER FLAT PLATES

The transition from laminar to turbulent flow depends on the *surface geometry*, *surface roughness*, *upstream velocity*, *surface temperature*, and the *type of fluid*, among other things, and is best characterized by the Reynolds number. The Reynolds number at a distance *x* from the leading edge of a flat plate is expressed as

$$\mathrm{Re}_x = \frac{\rho V x}{\mu} = \frac{V x}{\nu}$$



Laminar and turbulent regions of the boundary layer during flow over a flat plate.

A generally accepted value for the Critical Reynold number

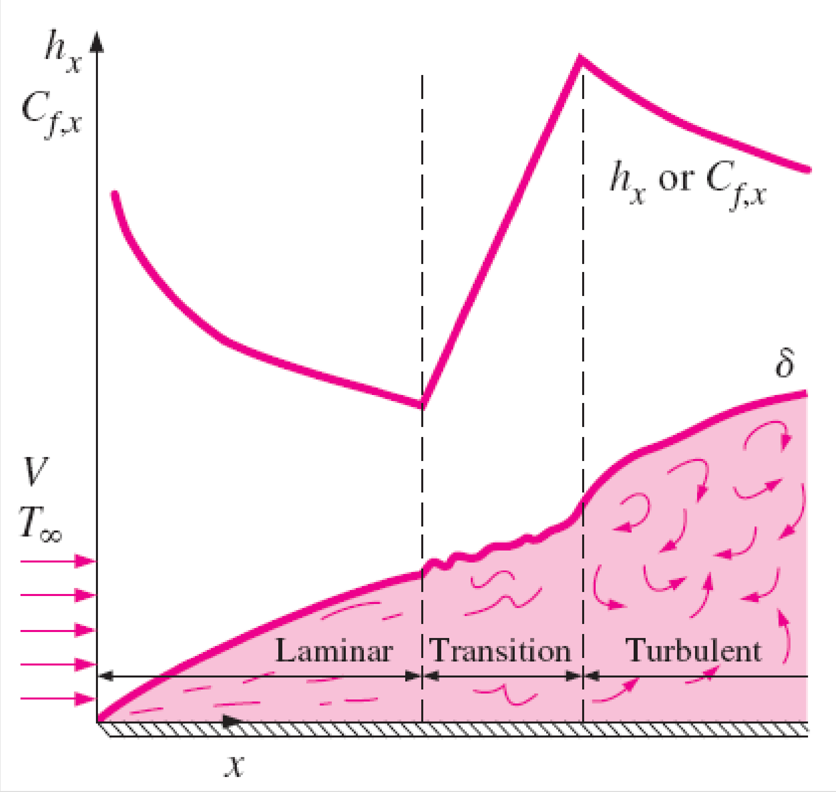
$$\mathrm{Re}_{cr} = \frac{\rho V x_{cr}}{\mu} = 5 \times 10^5$$

The actual value of the engineering critical Reynolds number for a flat plate may vary somewhat from 10^5 to 3×10^6, depending on the surface roughness, the turbulence level, and the variation of pressure along the surface.

The local Nusselt number at a location *x* for laminar flow over a flat plate may be obtained by solving the differential energy equation to be

Laminar: $$\mathrm{Nu}_x = \frac{h_x x}{k} = 0.332\ \mathrm{Re}_x^{0.5}\ \mathrm{Pr}^{1/3} \qquad \mathrm{Pr} > 0.6$$

Turbulent: $$\mathrm{Nu}_x = \frac{h_x x}{k} = 0.0296\ \mathrm{Re}_x^{0.8}\ \mathrm{Pr}^{1/3} \qquad 0.6 \le \mathrm{Pr} \le 60, \quad 5 \times 10^5 \le \mathrm{Re}_x \le 10^7$$



These relations are for *isothermal* and *smooth* surfaces

The local friction and heat transfer coefficients are higher in turbulent flow than they are in laminar flow.

Also, h_x reaches its highest values when the flow becomes fully turbulent, and then decreases by a factor of $x^{-0.2}$ in the flow direction.

The variation of the local friction and heat transfer coefficients for flow over a flat plate.

Nusselt numbers for average heat transfer coefficients

Laminar: $$\mathrm{Nu} = \frac{hL}{k} = 0.664\,\mathrm{Re}_L^{0.5}\,\mathrm{Pr}^{1/3} \qquad \mathrm{Re}_L < 5\times10^5$$

Turbulent: $$\mathrm{Nu} = \frac{hL}{k} = 0.037\,\mathrm{Re}_L^{0.8}\,\mathrm{Pr}^{1/3} \qquad 0.6 \le \mathrm{Pr} \le 60,\quad 5\times10^5 \le \mathrm{Re}_L \le 10^7$$

$$\mathrm{Nu} = \frac{hL}{k} = (0.037\,\mathrm{Re}_L^{0.8} - 871)\mathrm{Pr}^{1/3} \qquad 0.6 \le \mathrm{Pr} \le 60,\quad 5\times10^5 \le \mathrm{Re}_L \le 10^7$$

Laminar + turbulent

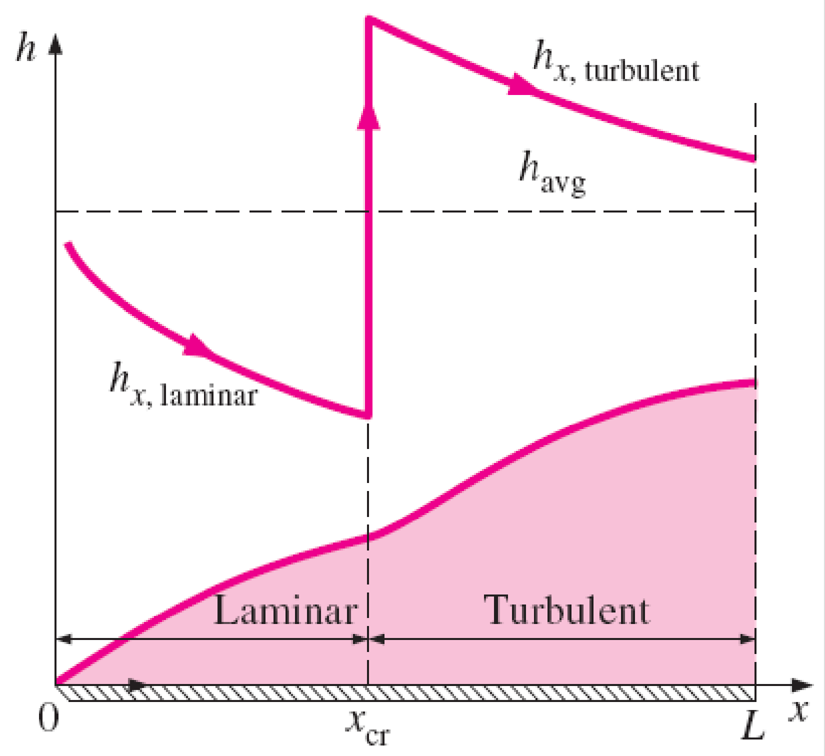
$$h = \frac{1}{L}\left(\int_0^{x_{cr}} h_{x,\,\mathrm{laminar}}\,dx + \int_{x_{cr}}^{L} h_{x,\,\mathrm{turbulent}}\,dx\right)$$

For liquid metals

$$\mathrm{Nu}_x = 0.565(\mathrm{Re}_x\,\mathrm{Pr})^{1/2} \qquad \mathrm{Pr} < 0.05$$

For all liquids, all Prandtl numbers

$$\mathrm{Nu}_x = \frac{h_x x}{k} = \frac{0.3387\,\mathrm{Pr}^{1/3}\,\mathrm{Re}_x^{1/2}}{[1 + (0.0468/\mathrm{Pr})^{2/3}]^{1/4}}$$



Graphical representation of the average heat transfer coefficient for a flat plate with combined laminar and turbulent flow.

Flat Plate with Unheated Starting Length

Local Nusselt numbers

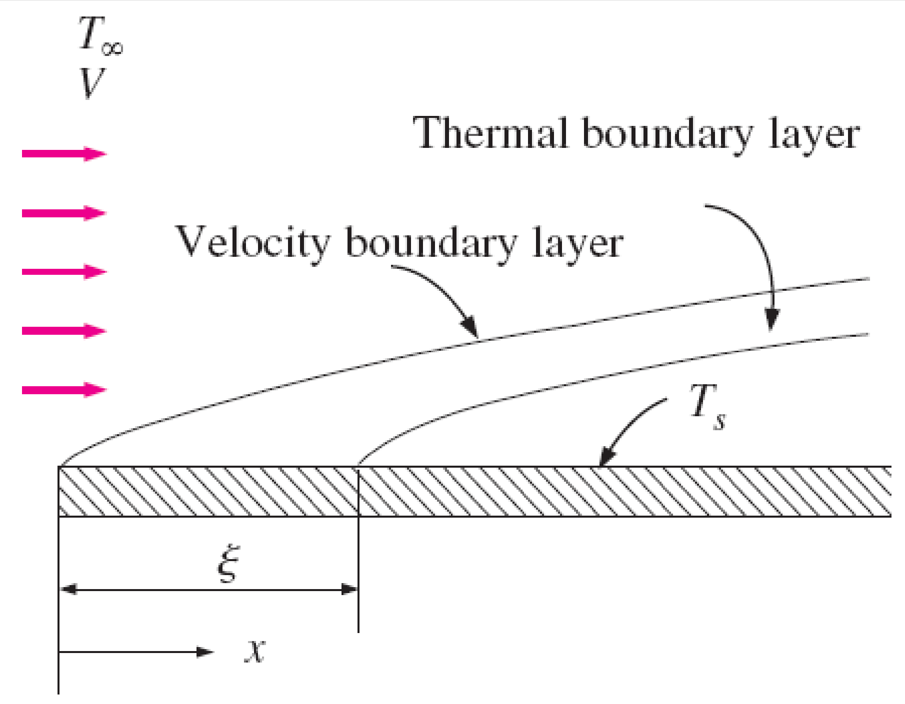
Laminar:
$$\text{Nu}_x = \frac{\text{Nu}_{x\,(\text{for } \xi=0)}}{[1-(\xi/x)^{3/4}]^{1/3}} = \frac{0.332\,\text{Re}_x^{0.5}\,\text{Pr}^{1/3}}{[1-(\xi/x)^{3/4}]^{1/3}}$$

Turbulent:
$$\text{Nu}_x = \frac{\text{Nu}_{x\,(\text{for } \xi=0)}}{[1-(\xi/x)^{9/10}]^{1/9}} = \frac{0.0296\,\text{Re}_x^{0.8}\,\text{Pr}^{1/3}}{[1-(\xi/x)^{9/10}]^{1/9}}$$

Average heat transfer coefficients

Laminar:
$$h = \frac{2[1-(\xi/x)^{3/4}]}{1-\xi/L}\,h_{x=L}$$

Turbulent:
$$h = \frac{5[1-(\xi/x)^{9/10}]}{4(1-\xi/L)}\,h_{x=L}$$



Flow over a flat plate with an unheated starting length.

Uniform Heat Flux

For a flat plate subjected to *uniform heat flux*

Laminar: $$\text{Nu}_x = 0.453\ \text{Re}_x^{0.5}\ \text{Pr}^{1/3}$$

Turbulent: $$\text{Nu}_x = 0.0308\ \text{Re}_x^{0.8}\ \text{Pr}^{1/3}$$

These relations give values that are 36 percent higher for laminar flow and 4 percent higher for turbulent flow relative to the isothermal plate case.

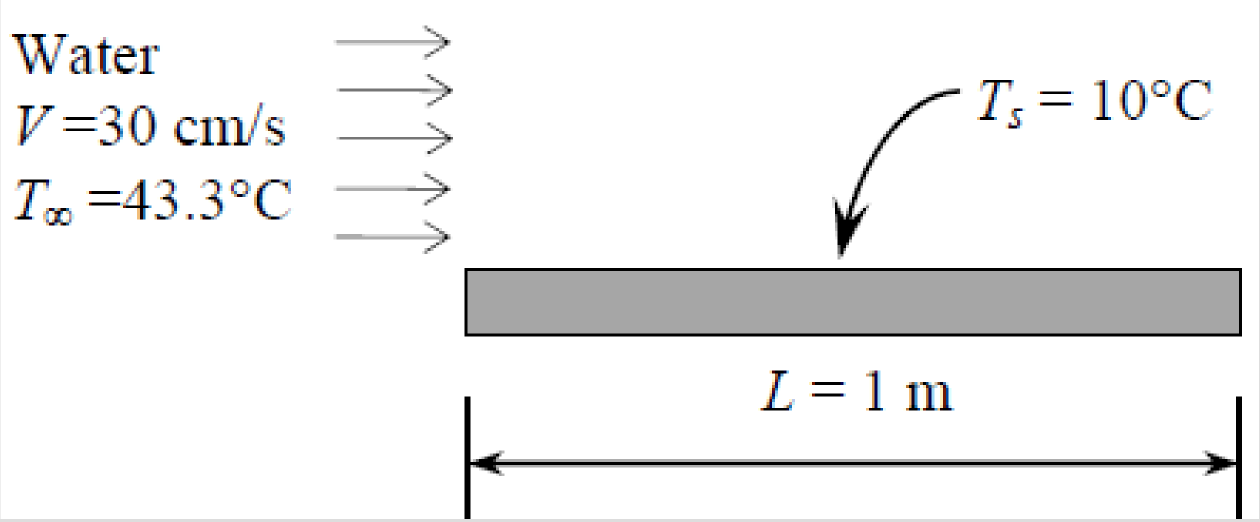
When heat flux is prescribed, the rate of heat transfer to or from the plate and the surface temperature at a distance *x* are determined from

$$\dot{Q} = \dot{q}_s A_s$$

$$\dot{q}_s = h_x[T_s(x) - T_\infty] \quad \rightarrow \quad T_s(x) = T_\infty + \frac{\dot{q}_s}{h_x}$$

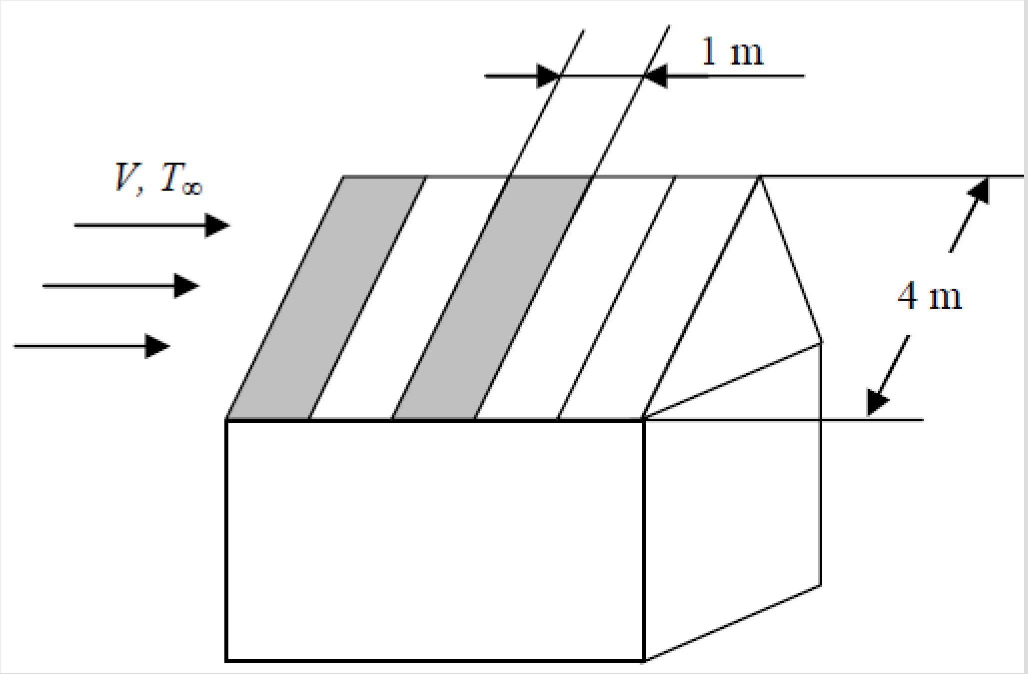
Example

Water at 43.3°C flows over a large plate at a velocity of 30 cm/s. The plate is 1.0 m long (in the flow direction), and its surface is maintained at uniform temperature of 10 °C. Calculate the steady rate of heat transfer per unit width of the plate.



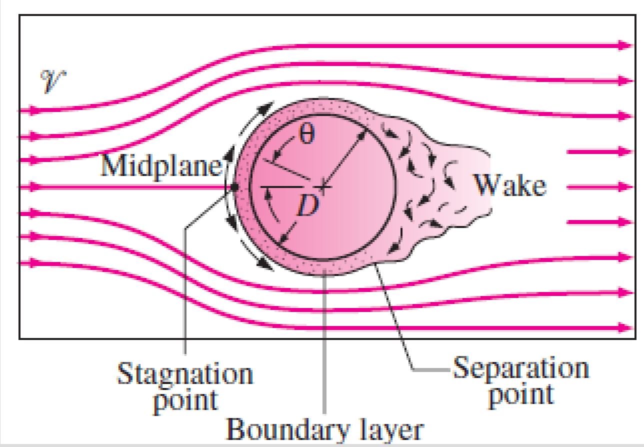
Example

Parallel plates form a solar collector that covers a roof, as shown in the figure. The plates are maintained at 15°C, while ambient air at 10°C flows over the roof with *V*=2 m/s. Determine the rate of convetive heat loss from the first plate.



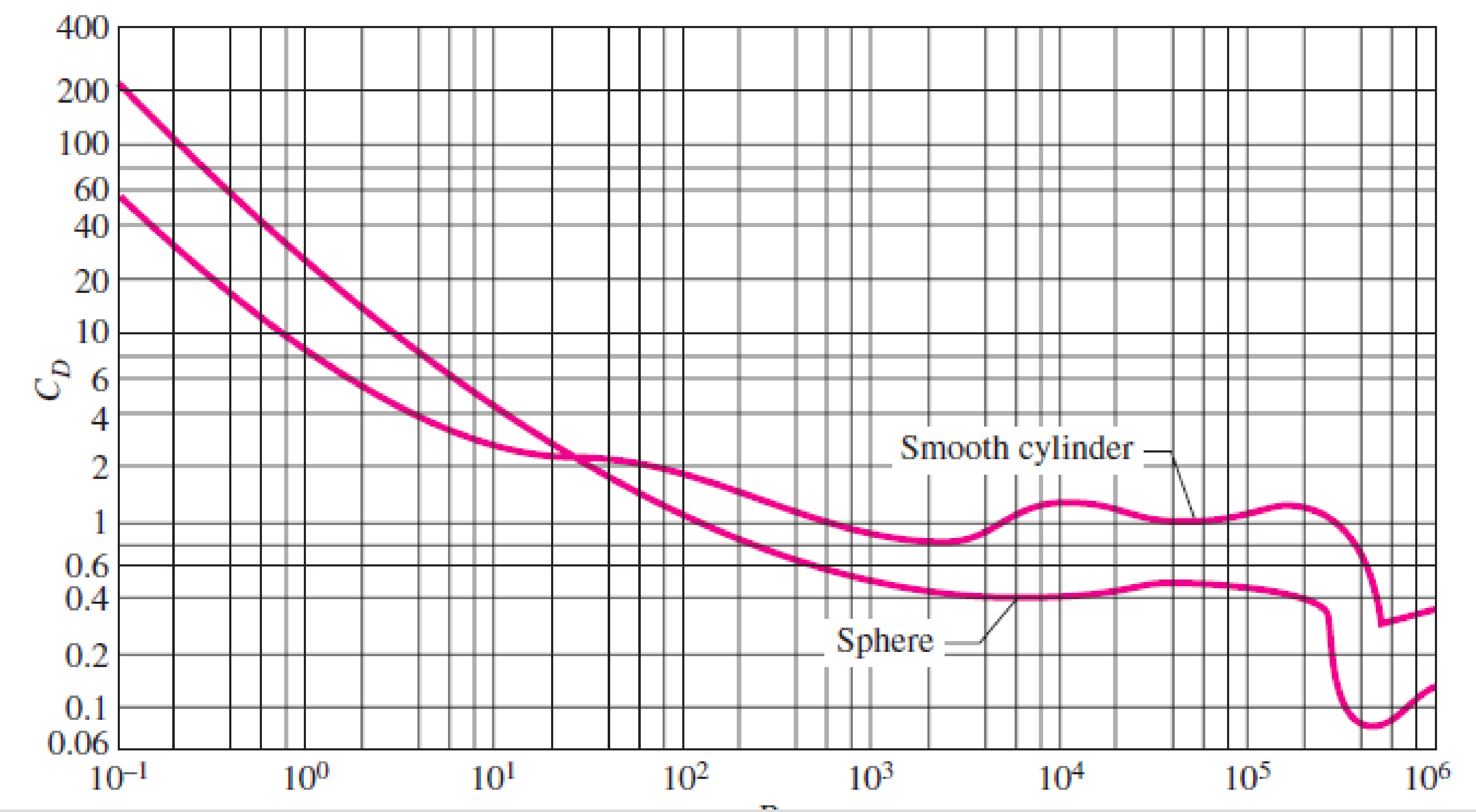
FLOW ACROSS CYLINDERS AND SPHERES

- Flows across cylinders and spheres, in general, involve *flow separation*, which is difficult to handle analytically.
- Flow across cylinders and spheres has been studied experimentally by numerous investigators, and several empirical correlations have been developed for the heat transfer coefficient.



The characteristic length for a circular cylinder or sphere is taken to be the *external diameter D*. Thus, the Reynolds number is defined as $Re = \mathcal{V}D/\nu$ where $\mathcal{V}$ is the uniform velocity of the fluid as it approaches the cylinder or sphere. The critical Reynolds number for flow across a circular cylinder or sphere is about $Re_{cr} \approx 2 \times 10^5$. That is, the boundary layer remains laminar for about $Re \lesssim 2 \times 10^5$ and becomes turbulent for $Re \gtrsim 2 \times 10^5$.

FLOW ACROSS CYLINDERS AND SPHERES



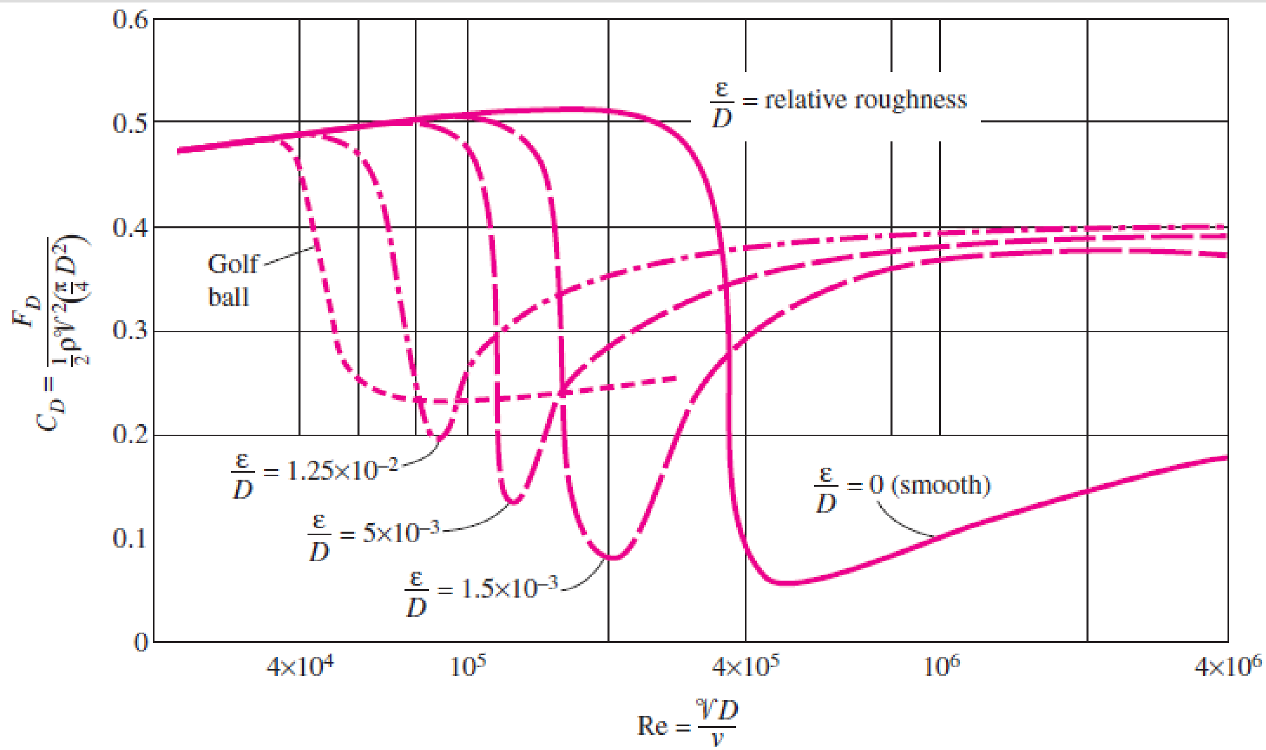
Average drag coefficient for cross flow over a smooth circular cylinder and a smooth sphere (from Schlichting, Ref. 10).

FLOW ACROSS CYLINDERS AND SPHERES

- For Re $\lesssim 1$, we have creeping flow, and the drag coefficient decreases with increasing Reynolds number. For a sphere, it is $C_D = 24/\text{Re}$. There is no flow separation in this regime.
- At about Re = 10, separation starts occurring on the rear of the body with vortex shedding starting at about Re $\approx$ 90. The region of separation increases with increasing Reynolds number up to about Re = 10^3. At this point, the drag is mostly (about 95 percent) due to pressure drag.
- In the moderate range of $10^3 < \text{Re} < 10^5$, the drag coefficient remains relatively constant.
- There is a sudden drop in the drag coefficient somewhere in the range of $10^5 < \text{Re} < 10^6$ (usually, at about 2×10^5). This large reduction in C_D is due to the flow in the boundary layer becoming *turbulent*, which moves the separation point further on the rear of the body, reducing the size of the wake and thus the magnitude of the pressure drag.

FLOW ACROSS CYLINDERS AND SPHERES

Effect of Surface Roughness



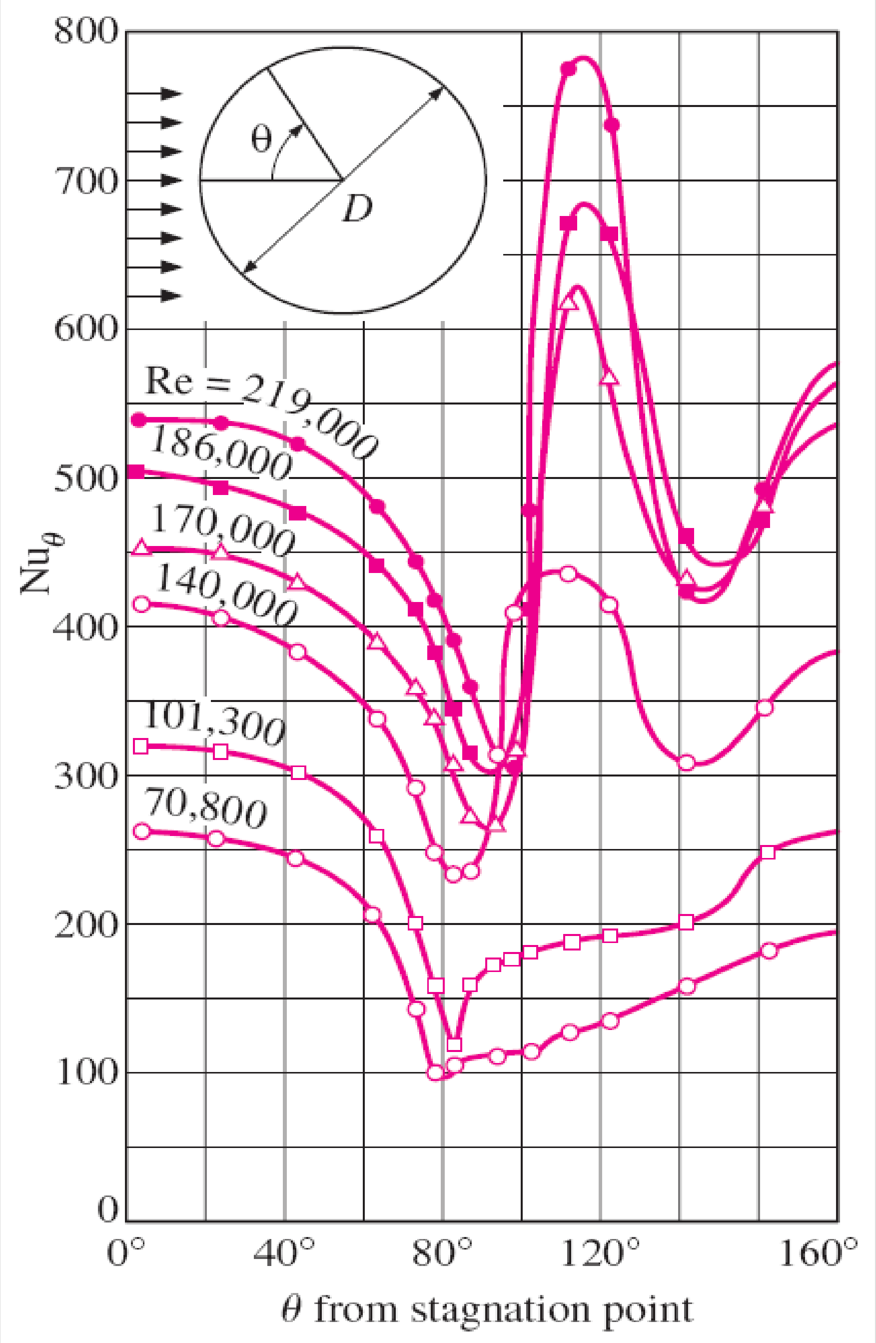
Re	C_D Smooth surface	C_D Rough surface, $\varepsilon/D = 0.0015$
10^5	0.5	0.1
10^6	0.1	0.4

Surface roughness may increase or decrease the drag coefficient of a spherical object, depending on the value of the Reynolds number.

The effect of surface roughness on the drag coefficient of a sphere (from Blevins, Ref. 1).

FLOW ACROSS CYLINDERS AND SPHERES

- **Heat Transfer Coefficient.**



Variation of the local heat transfer coefficient along the circumference of a circular cylinder in cross flow of air

For flow over a *cylinder*

$$\text{Nu}_{\text{cyl}} = \frac{hD}{k} = 0.3 + \frac{0.62\,\text{Re}^{1/2}\,\text{Pr}^{1/3}}{[1 + (0.4/\text{Pr})^{2/3}]^{1/4}}\left[1 + \left(\frac{\text{Re}}{282{,}000}\right)^{5/8}\right]^{4/5}$$

$\text{RePr} > 0.2$

The fluid properties are evaluated at the *film temperature* $T_f = \frac{1}{2}(T_\infty + T_s)$

For flow over a *sphere*

$$\text{Nu}_{\text{sph}} = \frac{hD}{k} = 2 + [0.4\,\text{Re}^{1/2} + 0.06\,\text{Re}^{2/3}]\,\text{Pr}^{0.4}\left(\frac{\mu_\infty}{\mu_s}\right)^{1/4}$$

$3.5 \le \text{Re} \le 80{,}000$ and $0.7 \le \text{Pr} \le 380$

The fluid properties are evaluated at the free-stream temperature T_∞, except for μ_s, which is evaluated at the surface temperature T_s.

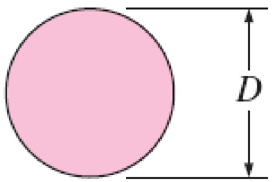
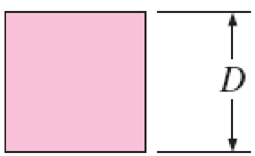
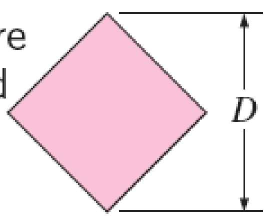
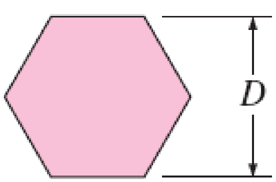
$$\text{Nu}_{\text{cyl}} = \frac{hD}{k} = C\,\text{Re}^m\,\text{Pr}^n$$

$n = \frac{1}{3}$

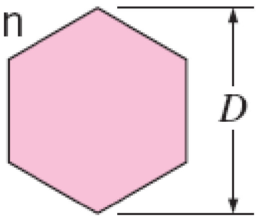
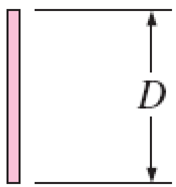
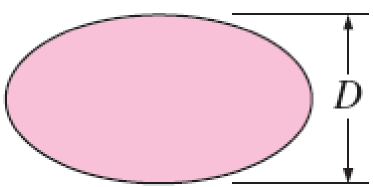
Constants *C* and *m* are given in the table.

The relations for cylinders above are for *single* cylinders or cylinders oriented such that the flow over them is not affected by the presence of others. They are applicable to *smooth* surfaces.

Empirical correlations for the average Nusselt number for forced convection over circular and noncircular cylinders in cross flow (from Zukauskas, 1972 and Jakob, 1949)

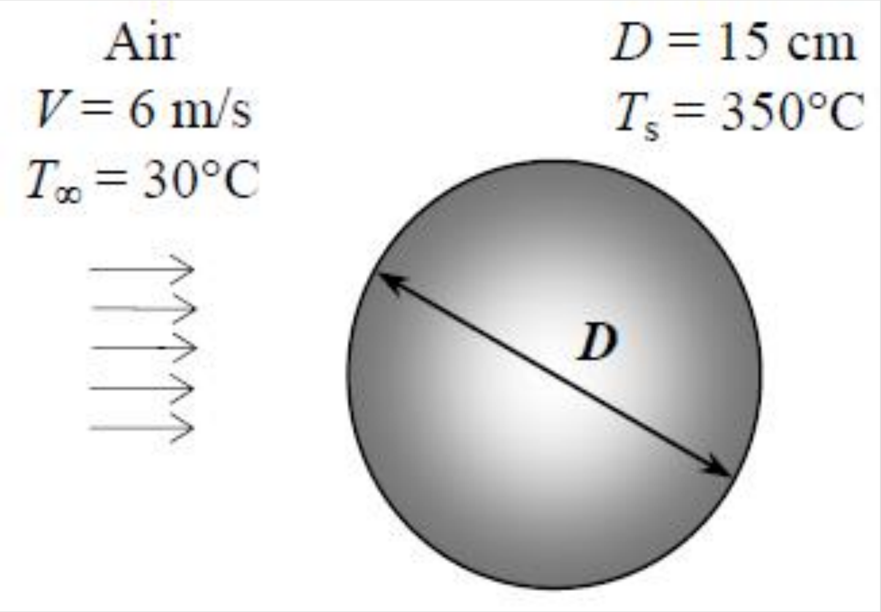
Cross-section of the cylinder	Fluid	Range of Re	Nusselt number
Circle 	Gas or liquid	0.4–4 4–40 40–4000 4000–40,000 40,000–400,000	$Nu = 0.989Re^{0.330}\ Pr^{1/3}$ $Nu = 0.911Re^{0.385}\ Pr^{1/3}$ $Nu = 0.683Re^{0.466}\ Pr^{1/3}$ $Nu = 0.193Re^{0.618}\ Pr^{1/3}$ $Nu = 0.027Re^{0.805}\ Pr^{1/3}$
Square 	Gas	5000–100,000	$Nu = 0.102Re^{0.675}\ Pr^{1/3}$
Square (tilted 45°) 	Gas	5000–100,000	$Nu = 0.246Re^{0.588}\ Pr^{1/3}$
Hexagon 	Gas	5000–100,000	$Nu = 0.153Re^{0.638}\ Pr^{1/3}$

Empirical correlations for the average Nusselt number for forced convection over circular and noncircular cylinders in cross flow (from Zukauskas, 1972 and Jakob, 1949)

Hexagon (tilted 45°) 	Gas	5000–19,500 19,500–100,000	$Nu = 0.160Re^{0.638}\ Pr^{1/3}$ $Nu = 0.0385Re^{0.782}\ Pr^{1/3}$
Vertical plate 	Gas	4000–15,000	$Nu = 0.228Re^{0.731}\ Pr^{1/3}$
Ellipse 	Gas	2500–15,000	$Nu = 0.248Re^{0.612}\ Pr^{1/3}$

Example

A stainless steel ball (ρ = 5055 kg/m^3, c_p = 480 J/Kg.°C) of diameter D=15 cm is removed from the oven at a uniform temperature of 350°C. The ball is then subjected to the flow of air at 1 atm pressure and 30°C with a velocity of 6 m/s. The surface temperature of the ball eventually drops to 250°C. Determine the average convection heat transfer coefficient during this process and estimate how long this procces has taken.



GENERAL CONSIDERATIONS FOR PIPE FLOW

Liquid or gas flow through pipes or ducts is commonly used in practice in heating and cooling applications. The fluid is forced to flow by a fan or pump through a conduit that is sufficiently long to accomplish the desired heat transfer.

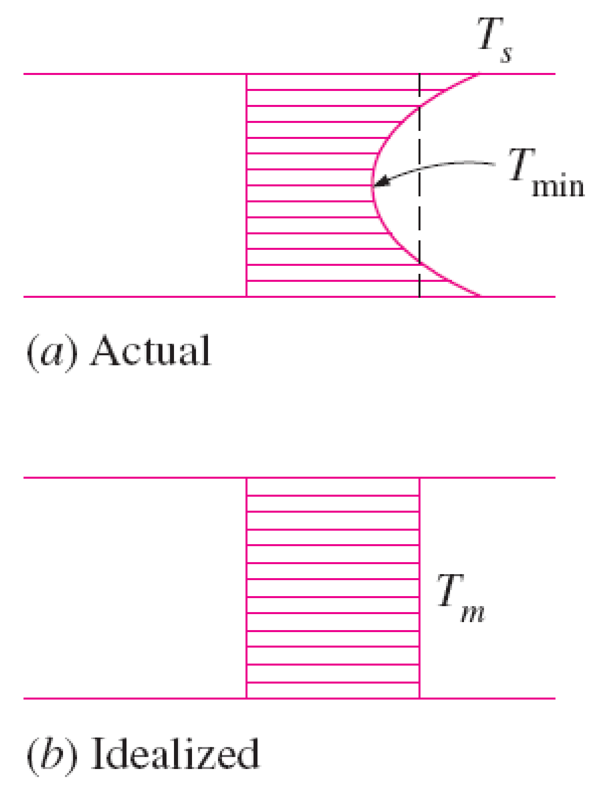
Transition from laminar to turbulent flow depends on the Reynolds number as well as the degree of disturbance of the flow by *surface roughness*, *pipe vibrations*, and the *fluctuations in the flow*.

The flow in a pipe is laminar for Re < 2300, fully turbulent for Re > 10,000, and transitional in between.

$$\mathrm{Re} = \frac{\rho V_{\mathrm{avg}} D}{\mu} = \frac{V_{\mathrm{avg}} D}{\nu}$$

$$\dot{E}_{\mathrm{fluid}} = \dot{m} c_p T_m = \int_{\dot{m}} c_p T(r) \delta \dot{m} = \int_{A_c} \rho c_p T(r) u(r) V dA_c$$

$$T_m = \frac{\int_{\dot{m}} c_p T(r) \delta \dot{m}}{\dot{m} c_p} = \frac{\int_0^R c_p T(r) \rho u(r) 2\pi r dr}{\rho V_{\mathrm{avg}} (\pi R^2) c_p} = \frac{2}{V_{\mathrm{avg}} R^2} \int_0^R T(r) u(r)\, r dr$$



Actual and idealized temperature profiles for flow in a tube (the rate at which energy is transported with the fluid is the same for both cases).

The fluid properties in internal flow are usually evaluated at the *bulk mean fluid temperature*, which is the arithmetic average of the mean temperatures at the inlet and the exit: $T_b = (T_{m,i} + T_{m,e})/2$

Thermal Entrance Region

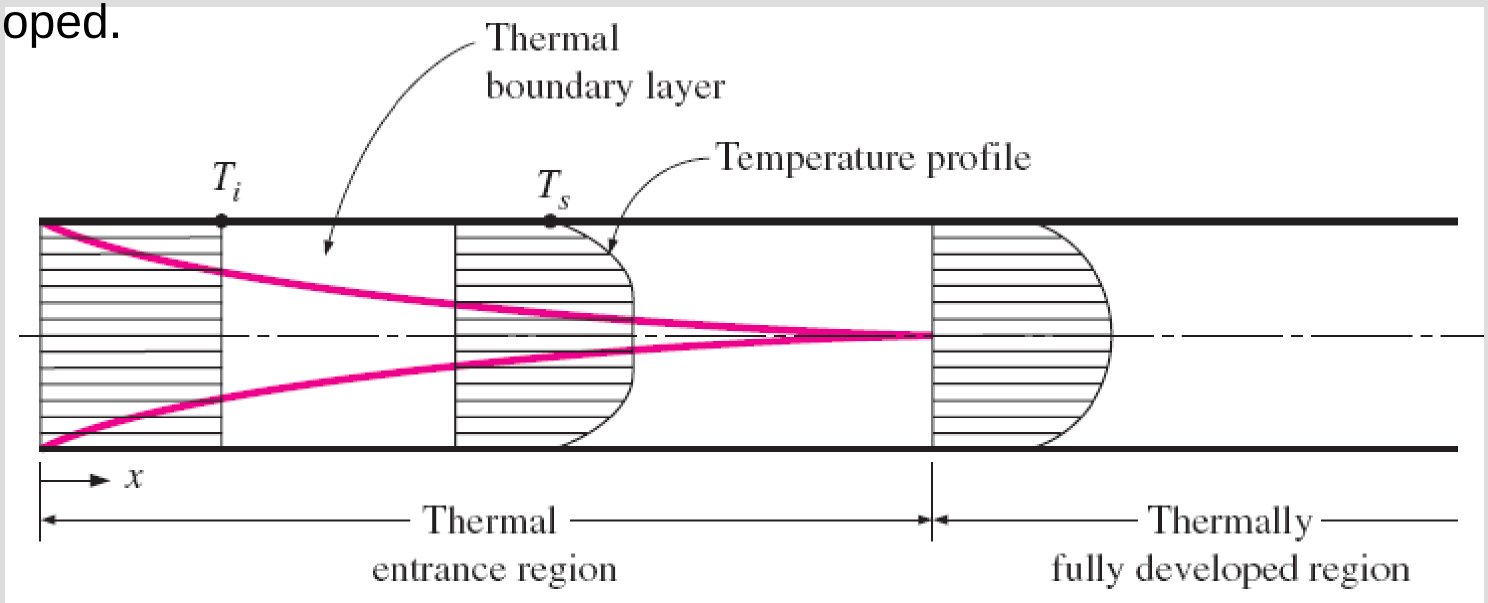
Thermal entrance region: The region of flow over which the thermal boundary layer develops and reaches the tube center.

Thermal entry length: The length of this region.

Thermally developing flow: Flow in the thermal entrance region. This is the region where the temperature profile develops.

Thermally fully developed region: The region beyond the thermal entrance region in which the dimensionless temperature profile remains unchanged.

Fully developed flow: The region in which the flow is both hydrodynamically and thermally developed.



The development of the thermal boundary layer in a tube.

Hydrodynamically fully developed:

$$\frac{\partial u(r, x)}{\partial x} = 0 \longrightarrow u = u(r)$$

Thermally fully developed:

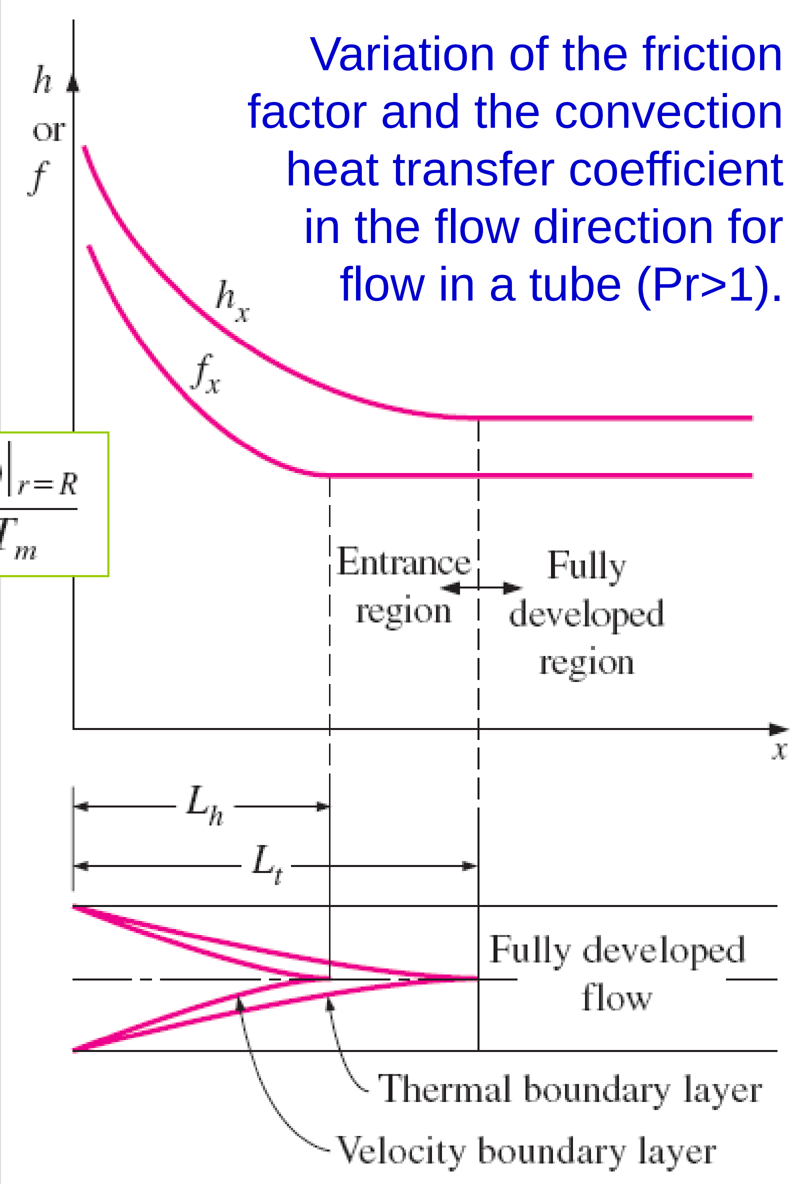
$$\frac{\partial}{\partial x}\left[\frac{T_s(x) - T(r, x)}{T_s(x) - T_m(x)}\right] = 0$$

$$\dot{q}_s = h_x(T_s - T_m) = k\left.\frac{\partial T}{\partial r}\right|_{r=R} \longrightarrow h_x = \frac{k(\partial T/\partial r)|_{r=R}}{T_s - T_m}$$

In the thermally fully developed region of a tube, the local convection coefficient is constant (does not vary with *x*).

Therefore, both *the friction* (*which is related to wall shear stress*) *and convection coefficients remain constant in the fully developed region of a tube.*

The pressure drop and heat flux are *higher* in the entrance regions of a tube, and the effect of the entrance region is always to *increase* the average friction factor and heat transfer coefficient for the entire tube.



Variation of the friction factor and the convection heat transfer coefficient in the flow direction for flow in a tube (Pr>1).

Entry Lengths

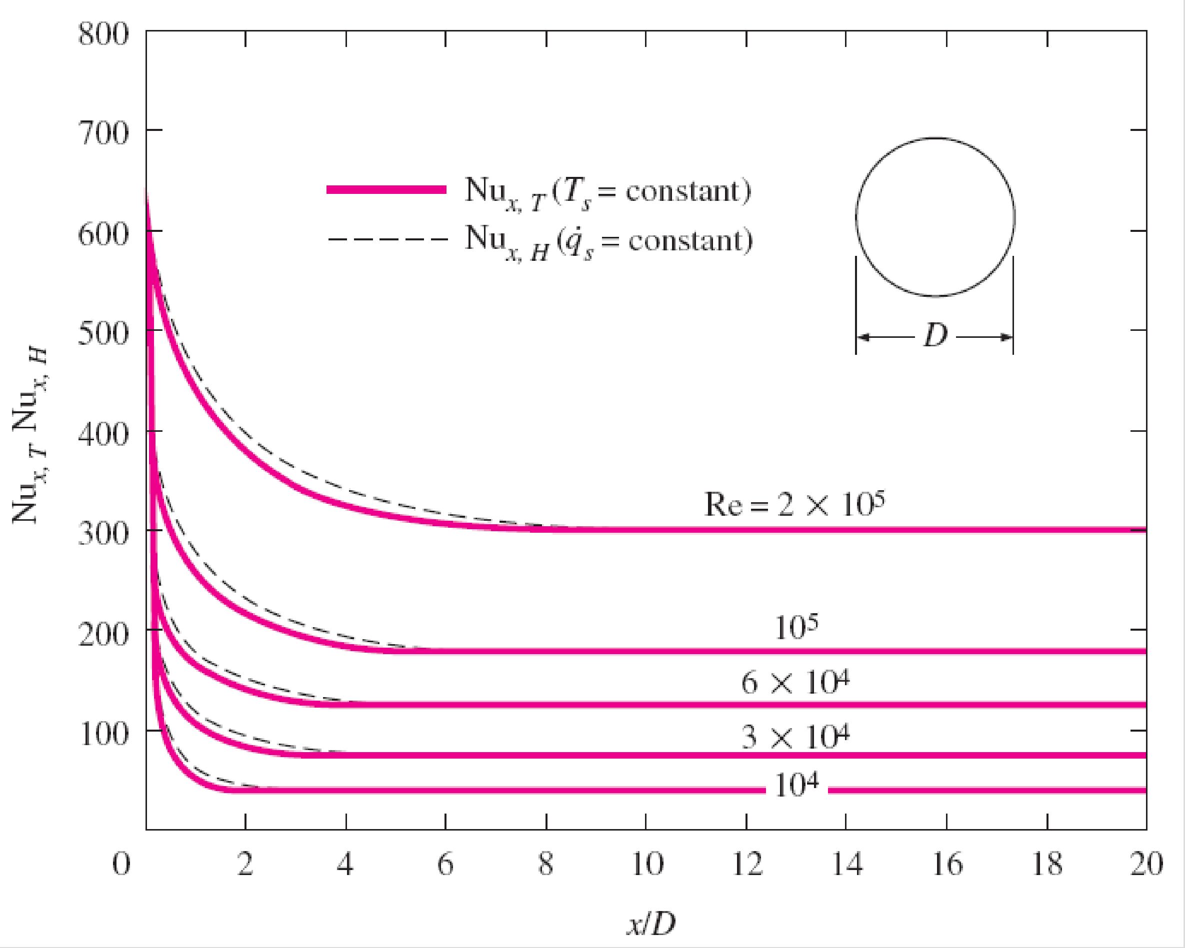
$$L_{h,\text{ laminar}} \approx 0.05\,\text{Re}\,D$$

$$L_{t,\text{ laminar}} \approx 0.05\,\text{Re}\,\text{Pr}\,D = \text{Pr}\,L_{h,\text{ laminar}}$$

$$L_{h,\text{ turbulent}} = 1.359 D\,\text{Re}^{1/4}$$

$$L_{h,\text{ turbulent}} \approx L_{t,\text{ turbulent}} \approx 10D$$

- The Nusselt numbers and thus *h* values are much higher in the entrance region.
- The Nusselt number reaches a constant value at a distance of less than 10 diameters, and thus the flow can be assumed to be fully developed for $x > 10D$.
- The Nusselt numbers for the uniform surface temperature and uniform surface heat flux conditions are identical in the fully developed regions, and nearly identical in the entrance regions.



Variation of local Nusselt number along a tube in turbulent flow for both uniform surface temperature and uniform surface heat flux.

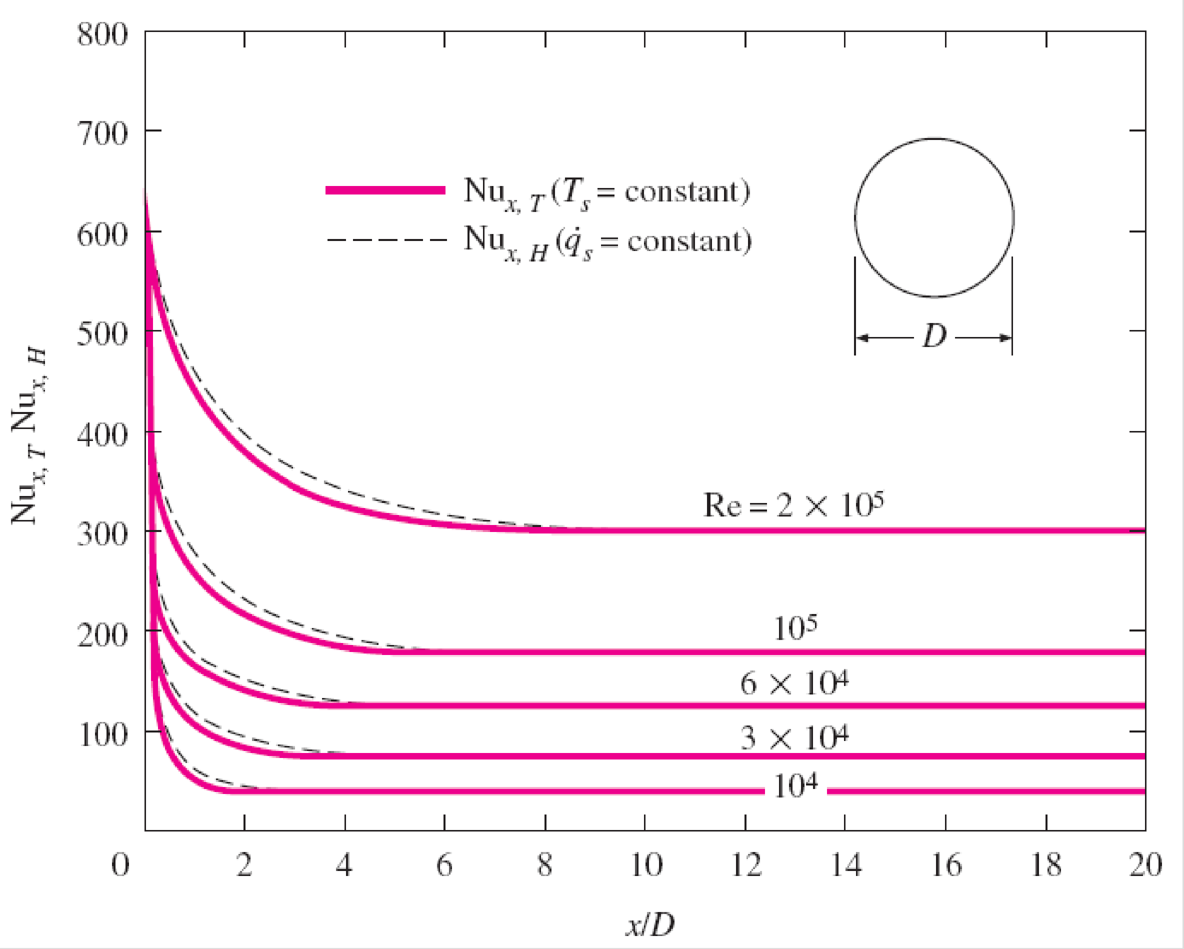
Entry Lengths

$$L_{h,\,\text{laminar}} \approx 0.05\,\text{Re}\,D$$

$$L_{t,\,\text{laminar}} \approx 0.05\,\text{Re}\,\text{Pr}\,D = \text{Pr}\,L_{h,\,\text{laminar}}$$

$$L_{h,\,\text{turbulent}} = 1.359 D\,\text{Re}^{1/4}$$

$$L_{h,\,\text{turbulent}} \approx L_{t,\,\text{turbulent}} \approx 10D$$



Variation of local Nusselt number along a tube in turbulent flow for both uniform surface temperature and uniform surface heat flux.

GENERAL THERMAL ANALYSIS

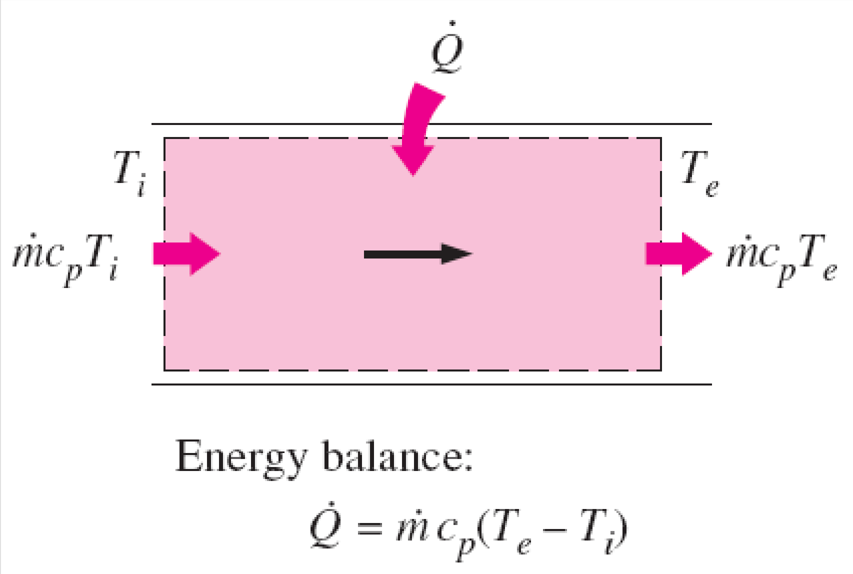
Rate of heat transfer

$$\dot{Q} = \dot{m} c_p (T_e - T_i) \qquad \text{(W)}$$

Surface heat flux

$$\dot{q}_s = h_x (T_s - T_m) \qquad (\text{W/m}^2)$$

h_x the *local* heat transfer coefficient



The heat transfer to a fluid flowing in a tube is equal to the increase in the energy of the fluid.

The thermal conditions at the surface can be approximated to be

constant surface temperature (T_s= const)

constant surface heat flux (q_s = const)

The constant surface temperature condition is realized when a phase change process such as boiling or condensation occurs at the outer surface of a tube.

The constant surface heat flux condition is realized when the tube is subjected to radiation or electric resistance heating uniformly from all directions.

We may have either T_s = constant or q_s = constant at the surface of a tube, but not both.

Constant Surface Heat Flux (q_s = constant)

Rate of heat transfer:

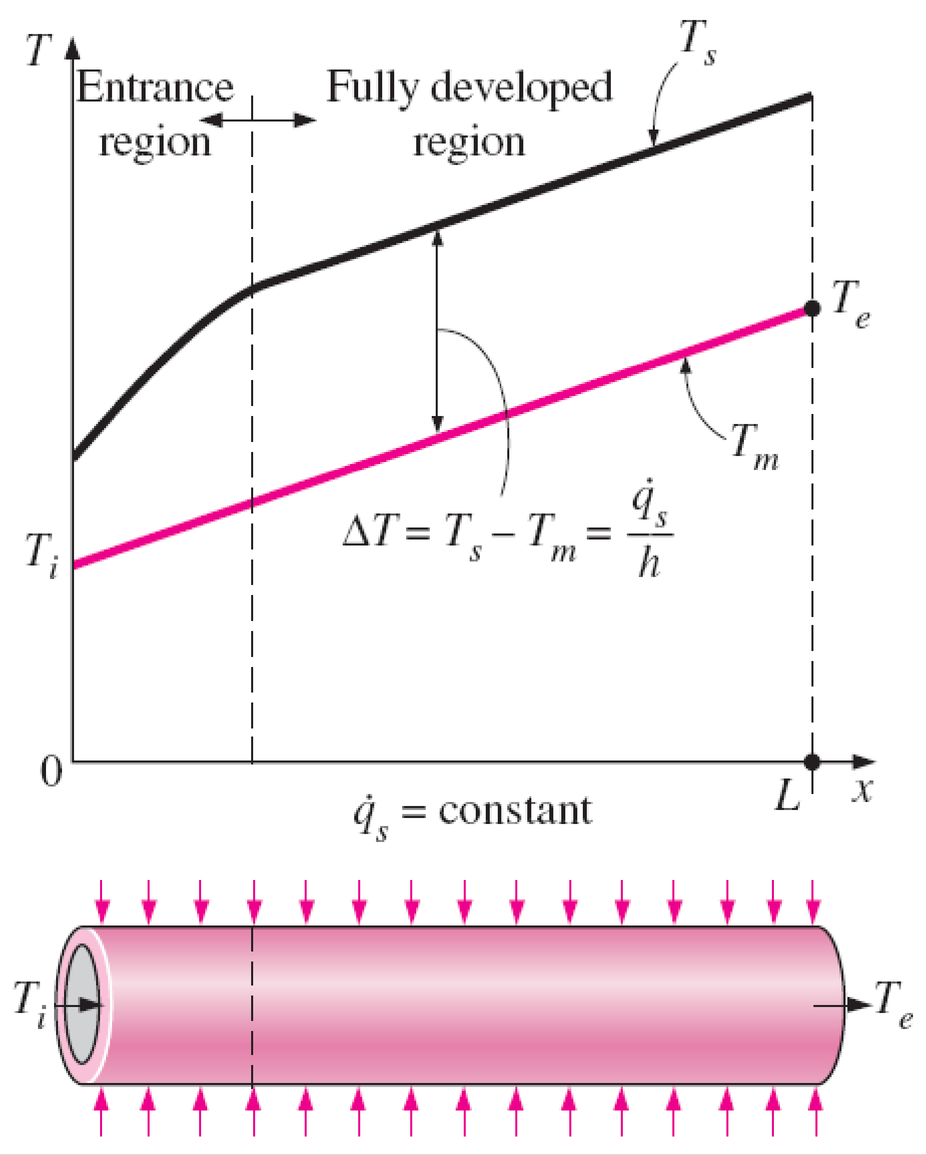
$$\dot{Q} = \dot{q}_s A_s = \dot{m} c_p (T_e - T_i) \qquad \text{(W)}$$

Mean fluid temperature at the tube exit:

$$T_e = T_i + \frac{\dot{q}_s A_s}{\dot{m} c_p}$$

Surface temperature:

$$\dot{q}_s = h(T_s - T_m) \longrightarrow T_s = T_m + \frac{\dot{q}_s}{h}$$



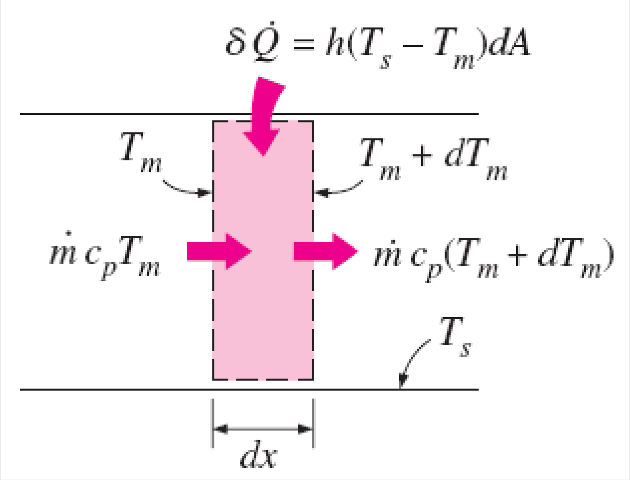
Variation of the *tube surface* and the *mean fluid* temperatures along the tube for the case of constant surface heat flux.

$$\dot{m} c_p \, dT_m = \dot{q}_s (p\,dx) \longrightarrow \frac{dT_m}{dx} = \frac{\dot{q}_s p}{\dot{m} c_p} = \text{constant}$$

$$\frac{dT_m}{dx} = \frac{dT_s}{dx}$$

$$\frac{\partial}{\partial x}\left(\frac{T_s - T}{T_s - T_m}\right) = 0 \longrightarrow \frac{1}{T_s - T_m}\left(\frac{\partial T_s}{\partial x} - \frac{\partial T}{\partial x}\right) = 0$$

$$\longrightarrow \frac{\partial T}{\partial x} = \frac{dT_s}{dx}$$

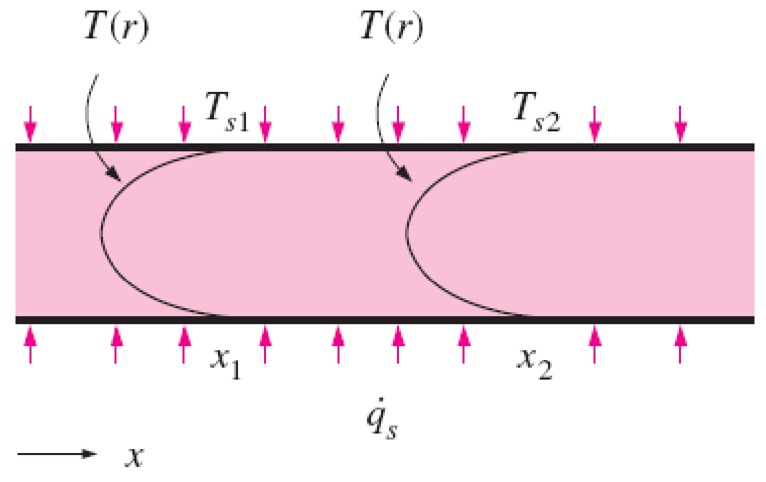


Energy interactions for a differential control volume in a tube.

$$\frac{\partial T}{\partial x} = \frac{dT_s}{dx} = \frac{dT_m}{dx} = \frac{\dot{q}_s p}{\dot{m} c_p} = \text{constant}$$

Circular tube:

$$\frac{\partial T}{\partial x} = \frac{dT_s}{dx} = \frac{dT_m}{dx} = \frac{2\dot{q}_s}{\rho V_{avg} c_p R} = \text{constant}$$



The shape of the temperature profile remains unchanged in the fully developed region of a tube subjected to constant surface heat flux.

Constant Surface Temperature (T_s = constant)

Rate of heat transfer to or from a fluid flowing in a tube

$$\dot{Q} = hA_s \Delta T_{\text{avg}} = hA_s (T_s - T_m)_{\text{avg}} \qquad \text{(W)}$$

Two suitable ways of expressing ΔT_{avg}

- arithmetic mean temperature difference
- logarithmic mean temperature difference

Arithmetic mean temperature difference

$$\Delta T_{\text{avg}} \approx \Delta T_{\text{am}} = \frac{\Delta T_i + \Delta T_e}{2} = \frac{(T_s - T_i) + (T_s - T_e)}{2} = T_s - \frac{T_i + T_e}{2} = T_s - T_b$$

Bulk mean fluid temperature: $T_b = (T_i + T_e)/2$

By using arithmetic mean temperature difference, we assume that the mean fluid temperature varies linearly along the tube, which is hardly ever the case when T_s = constant.

This simple approximation often gives acceptable results, but not always.

Therefore, we need a better way to evaluate ΔT_{avg}.

$$\dot{m} c_p \, dT_m = h(T_s - T_m) dA_s$$

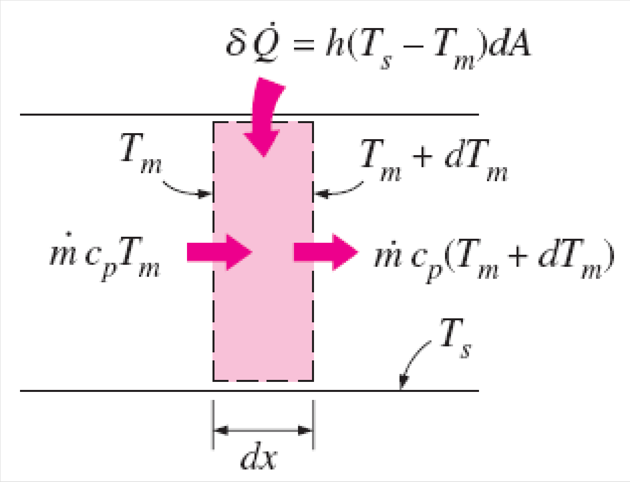
$$dA_s = p dx \qquad dT_m = -d(T_s - T_m)$$

$$\frac{d(T_s - T_m)}{T_s - T_m} = -\frac{hp}{\dot{m} c_p} dx$$

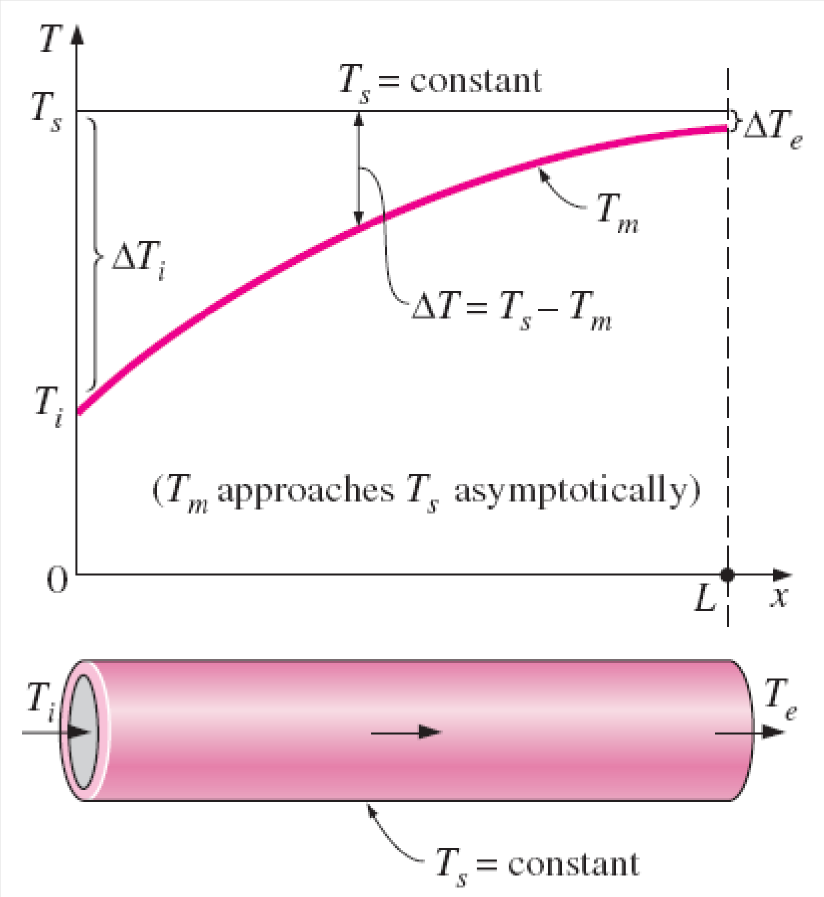
Integrating from $x = 0$ (tube inlet, $T_m = T_i$) to $x = L$ (tube exit, $T_m = T_e$)

$$\ln \frac{T_s - T_e}{T_s - T_i} = -\frac{hA_s}{\dot{m} c_p}$$

$$T_e = T_s - (T_s - T_i) \exp(-hA_s/\dot{m} c_p)$$



Energy interactions for a differential control volume in a tube.



The variation of the *mean fluid* temperature along the tube for the case of constant temperature.

$$\dot{Q} = hA_s\Delta T_{\ln}$$

$$\Delta T_{\ln} = \frac{T_i - T_e}{\ln[(T_s - T_e)/(T_s - T_i)]} = \frac{\Delta T_e - \Delta T_i}{\ln(\Delta T_e/\Delta T_i)}$$

logarithmic mean temperature difference

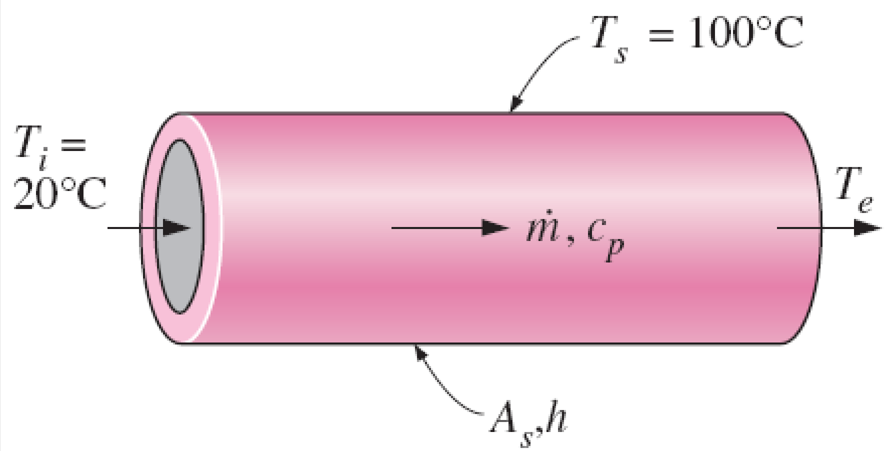
NTU: *Number of transfer units.* A measure of the effectiveness of the heat transfer systems.

For NTU = 5, $T_e = T_s$, and the limit for heat transfer is reached.

A small value of NTU indicates more opportunities for heat transfer.

$\Delta T_{\ln}$ is an *exact* representation of the *average temperature difference* between the fluid and the surface.

When ΔT_e differs from ΔT_i by no more than 40 percent, the error in using the arithmetic mean temperature difference is less than 1 percent.



NTU = $hA_s / \dot{m}c_p$	T_e, °C
0.01	20.8
0.05	23.9
0.10	27.6
0.50	51.5
1.00	70.6
5.00	99.5
10.00	100.0

An NTU greater than 5 indicates that the fluid flowing in a tube will reach the surface temperature at the exit regardless of the inlet temperature.

LAMINAR FLOW IN TUBES

$$\dot{m}c_pT_x - \dot{m}c_pT_{x+dx} + \dot{Q}_r - \dot{Q}_{r+dr} = 0$$

$$\dot{m} = \rho u A_c = \rho u(2\pi r dr)$$

$$\rho c_p u \frac{T_{x+dx} - T_x}{dx} = -\frac{1}{2\pi r dx}\frac{\dot{Q}_{r+dr} - \dot{Q}_r}{dr}$$

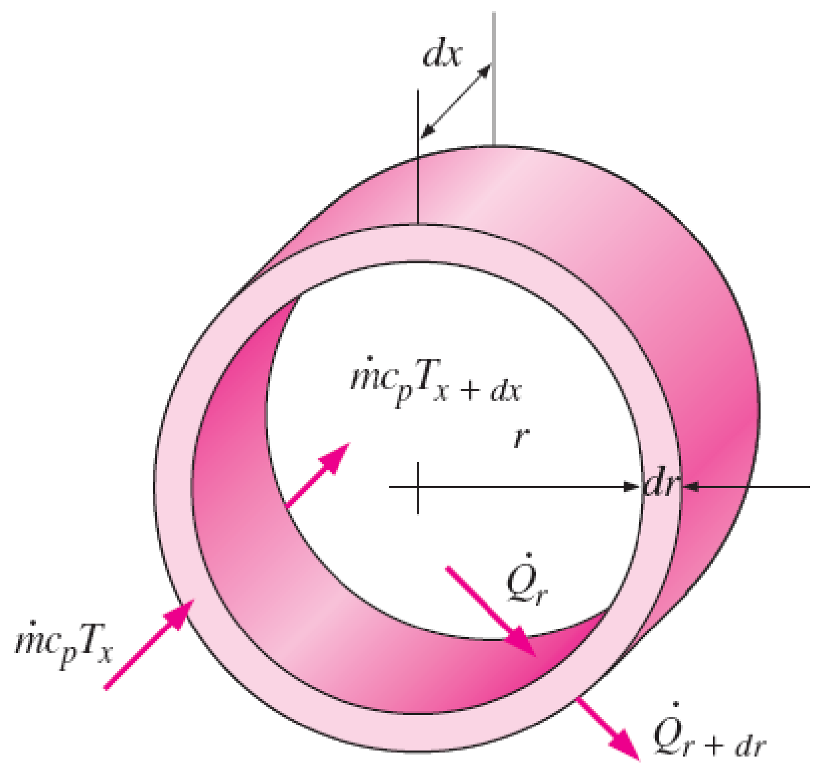
$$u\frac{\partial T}{\partial x} = -\frac{1}{2\rho c_p \pi r dx}\frac{\partial \dot{Q}}{\partial r}$$

$$\frac{\partial \dot{Q}}{\partial r} = \frac{\partial}{\partial r}\left(-k2\pi r dx\frac{\partial T}{\partial r}\right) = -2\pi k dx\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$$

$$\alpha = k/\rho c_p$$

$$u\frac{\partial T}{\partial x} = \frac{\alpha}{r}\frac{\partial}{dr}\left(r\frac{\partial T}{\partial r}\right)$$

The rate of net energy transfer to the control volume by mass flow is equal to the net rate of heat conduction in the radial direction.



The differential volume element used in the derivation of energy balance relation.

Constant Surface Heat Flux

$$\frac{\partial T}{\partial x} = \frac{dT_s}{dx} = \frac{dT_m}{dx} = \frac{2\dot{q}_s}{\rho V_{\text{avg}} c_p R} = \text{constant}$$

$$\frac{4\dot{q}_s}{kR}\left(1 - \frac{r^2}{R^2}\right) = \frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)$$

$$T = \frac{\dot{q}_s}{kR}\left(r^2 - \frac{r^4}{4R^2}\right) + C_1 r + C_2$$

Applying the boundary conditions $\partial T/\partial x = 0$ at $r = 0$ (because of symmetry) and $T = T_s$ at $r = R$

$$T = T_s - \frac{\dot{q}_s R}{k}\left(\frac{3}{4} - \frac{r^2}{R^2} + \frac{r^4}{4R^4}\right)$$

$$T_m = T_s - \frac{11}{24}\frac{\dot{q}_s R}{k}$$

$$\dot{q}_s = h(T_s - T_m)$$

$$h = \frac{24}{11}\frac{k}{R} = \frac{48}{11}\frac{k}{D} = 4.36\frac{k}{D}$$

Circular tube, laminar ($\dot{q}_s$ = constant):

$$\text{Nu} = \frac{hD}{k} = 4.36$$

Therefore, for fully developed laminar flow in a circular tube subjected to constant surface heat flux, the Nusselt number is a constant.

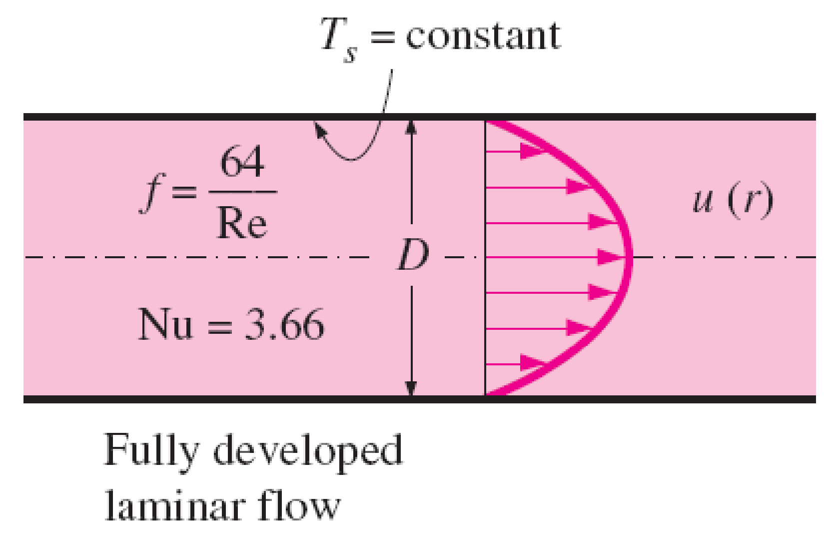
There is no dependence on the Reynolds or the Prandtl numbers.

Constant Surface Temperature

Circular tube, laminar (T_s = constant): $$\text{Nu} = \frac{hD}{k} = 3.66$$

The thermal conductivity *k* for use in the Nu relations should be evaluated at the bulk mean fluid temperature.

For laminar flow, the effect of *surface roughness* on the friction factor and the heat transfer coefficient is negligible.



In laminar flow in a tube with constant surface temperature, both the *friction factor* and the *heat transfer coefficient* remain constant in the fully developed region.

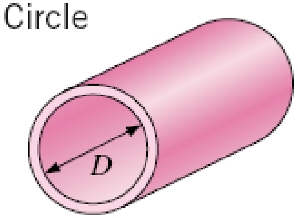
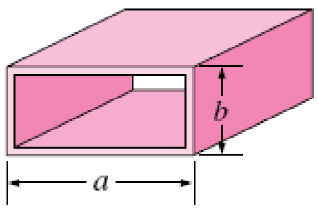
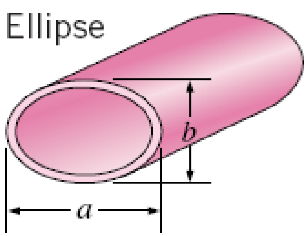
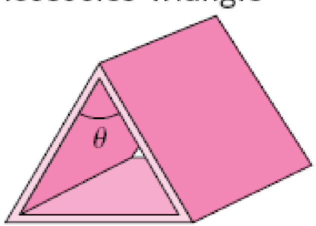
Laminar Flow in Noncircular Tubes

Nusselt number relations are given in the table for *fully developed laminar flow* in tubes of various cross sections.

The Reynolds and Nusselt numbers for flow in these tubes are based on the hydraulic diameter $D_h = 4A_c/p$,

Once the Nusselt number is available, the convection heat transfer coefficient is determined from $h = k\text{Nu}/D_h$.

Nusselt number and friction factor for fully developed laminar flow in tubes of various cross sections ($D_h = 4A_c/p$, $\mathrm{Re} = V_{avg}D_h/\nu$, and $\mathrm{Nu} = hD_h/k$)

Tube Geometry	a/b or θ°	Nusselt Number T_s = Const.	Nusselt Number $\dot{q}_s$ = Const.
Circle 	—	3.66	4.36
Rectangle 	a/b		
	1	2.98	3.61
	2	3.39	4.12
	3	3.96	4.79
	4	4.44	5.33
	6	5.14	6.05
	8	5.60	6.49
	∞	7.54	8.24
Ellipse 	a/b		
	1	3.66	4.36
	2	3.74	4.56
	4	3.79	4.88
	8	3.72	5.09
	16	3.65	5.18
Isosceles Triangle 	θ		
	10°	1.61	2.45
	30°	2.26	2.91
	60°	2.47	3.11
	90°	2.34	2.98
	120°	2.00	2.68

Developing Laminar Flow in the Entrance Region

For a circular tube of length *L* subjected to constant surface temperature, the average Nusselt number for the *thermal entrance region*:

Entry region, laminar:

$$\mathrm{Nu} = 3.66 + \frac{0.065\,(D/L)\,\mathrm{Re}\,\mathrm{Pr}}{1 + 0.04[(D/L)\,\mathrm{Re}\,\mathrm{Pr}]^{2/3}}$$

The average Nusselt number is larger at the entrance region, and it approaches asymptotically to the fully developed value of 3.66 as $L \to \infty$.

When the difference between the surface and the fluid temperatures is large, it may be necessary to account for the variation of viscosity with temperature:

$$\mathrm{Nu} = 1.86\left(\frac{\mathrm{Re}\,\mathrm{Pr}\,D}{L}\right)^{1/3}\left(\frac{\mu_b}{\mu_s}\right)^{0.14}$$

All properties are evaluated at the bulk mean fluid temperature, except for μ_s, which is evaluated at the surface temperature.

The average Nusselt number for the thermal entrance region of flow between *isothermal parallel plates* of length *L* is

Entry region, laminar:

$$\mathrm{Nu} = 7.54 + \frac{0.03\,(D_h/L)\,\mathrm{Re}\,\mathrm{Pr}}{1 + 0.016[(D_h/L)\,\mathrm{Re}\,\mathrm{Pr}]^{2/3}}$$

$$\mathrm{Re} \leq 2800$$

TURBULENT FLOW IN TUBES

Smooth tubes: $f = (0.790 \ln Re - 1.64)^{-2}$ $\quad 3000 < Re < 5 \times 10^6$

First Petukhov equation

$Nu = 0.125 f Re Pr^{1/3}$ *Chilton–Colburn analogy*

$f = 0.184 Re^{-0.2}$

$$Nu = 0.023 Re^{0.8} Pr^{1/3} \quad \begin{pmatrix} 0.7 \le Pr \le 160 \\ Re > 10{,}000 \end{pmatrix}$$

Colburn equation

$Nu = 0.023 Re^{0.8} Pr^n$ *Dittus–Boelter equation*

$n = 0.4$ for *heating* and 0.3 for *cooling*

When the variation in properties is large due to a large temperature difference

$$Nu = 0.027 Re^{0.8} Pr^{1/3} \left(\frac{\mu}{\mu_s}\right)^{0.14} \quad \begin{pmatrix} 0.7 \le Pr \le 17{,}600 \\ Re \ge 10{,}000 \end{pmatrix}$$

All properties are evaluated at T_b except μ_s, which is evaluated at T_s.

$$\text{Nu} = \frac{(f/8)\,\text{Re}\,\text{Pr}}{1.07 + 12.7(f/8)^{0.5}(\text{Pr}^{2/3} - 1)} \qquad \left(\begin{array}{l} 0.5 \le \text{Pr} \le 2000 \\ 10^4 < \text{Re} < 5 \times 10^6 \end{array}\right)$$

Second Petukhov equation

$$\text{Nu} = \frac{(f/8)(\text{Re} - 1000)\,\text{Pr}}{1 + 12.7(f/8)^{0.5}(\text{Pr}^{2/3} - 1)} \qquad \left(\begin{array}{l} 0.5 \le \text{Pr} \le 2000 \\ 3 \times 10^3 < \text{Re} < 5 \times 10^6 \end{array}\right)$$

Gnielinski relation

Liquid metals, T_s = constant: $\text{Nu} = 4.8 + 0.0156\,\text{Re}^{0.85}\,\text{Pr}_s^{0.93}$

Liquid metals, $\dot{q}_s$ = constant: $\text{Nu} = 6.3 + 0.0167\,\text{Re}^{0.85}\,\text{Pr}_s^{0.93}$

$(0.004 < \text{Pr} < 0.01)$ $\quad 10^4 < \text{Re} < 10^6$

In turbulent flow, wall roughness increases the heat transfer coefficient *h* by a factor of 2 or more. The convection heat transfer coefficient for rough tubes can be calculated approximately from *Gnielinski relation* or *Chilton–Colburn analogy* by using the friction factor determined from the *Moody chart* or the *Colebrook equation*.

The relations above are not very sensitive to the *thermal conditions* at the tube surfaces and can be used for both T_s = constant and q_s = constant.

Equivalent roughness values for new commercial pipes*

	Roughness, ε	
Material	ft	mm
Glass, plastic	0 (smooth)	
Concrete	0.003–0.03	0.9–9
Wood stave	0.0016	0.5
Rubber, smoothed	0.000033	0.01
Copper or brass tubing	0.000005	0.0015
Cast iron	0.00085	0.26
Galvanized iron	0.0005	0.15
Wrought iron	0.00015	0.046
Stainless steel	0.000007	0.002
Commercial steel	0.00015	0.045

*The uncertainty in these values can be as much as ±60 percent.

Developing Turbulent Flow in the Entrance Region

The entry lengths for turbulent flow are typically short, often just 10 tube diameters long, and thus the Nusselt number determined for fully developed turbulent flow can be used approximately for the entire tube.

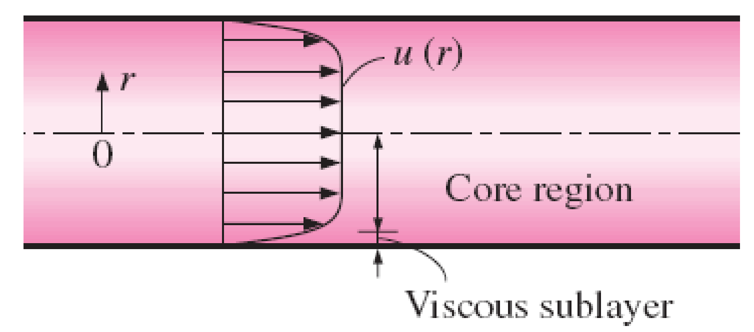
This simple approach gives reasonable results for pressure drop and heat transfer for long tubes and conservative results for short ones.

Correlations for the friction and heat transfer coefficients for the entrance regions are available in the literature for better accuracy.

Turbulent Flow in Noncircular Tubes

Pressure drop and heat transfer characteristics of turbulent flow in tubes are dominated by the very thin viscous sublayer next to the wall surface, and the shape of the core region is not of much significance.

The turbulent flow relations given above for circular tubes can also be used for noncircular tubes with reasonable accuracy by replacing the diameter *D* in the evaluation of the Reynolds number by the hydraulic diameter $D_h = 4A_c/p$.

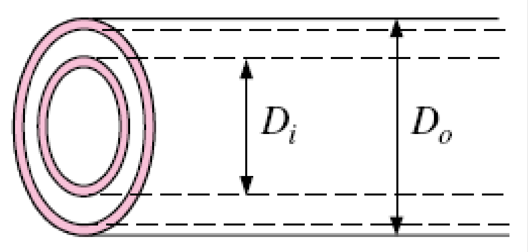


In turbulent flow, the velocity profile is nearly a straight line in the core region, and any significant velocity gradients occur in the viscous sublayer.

Flow through Tube Annulus

$$D_h = \frac{4A_c}{p} = \frac{4\pi(D_o^2 - D_i^2)/4}{\pi(D_o + D_i)} = D_o - D_i$$

The hydraulic diameter of annulus



For laminar flow, the convection coefficients for the inner and the outer surfaces are determined from

$$\text{Nu}_i = \frac{h_i D_h}{k} \quad \text{and} \quad \text{Nu}_o = \frac{h_o D_h}{k}$$

Tube surfaces are often *roughened, corrugated,* or *finned* in order to *enhance* convection heat transfer.

For fully developed turbulent flow, h_i and h_o are approximately equal to each other, and the tube annulus can be treated as a noncircular duct with a hydraulic diameter of $D_h = D_o - D_i$.

The Nusselt number can be determined from a suitable turbulent flow relation such as the Gnielinski equation. To improve the accuracy, Nusselt number can be multiplied by the following correction factors when one of the tube walls is adiabatic and heat transfer is through the other wall:

$$F_i = 0.86\left(\frac{D_i}{D_o}\right)^{-0.16} \quad \text{(outer wall adiabatic)}$$

$$F_o = 0.86\left(\frac{D_i}{D_o}\right)^{-0.16} \quad \text{(inner wall adiabatic)}$$

Nusselt number for fully developed laminar flow in an annulus with one surface isothermal and the other adiabatic (Kays and Perkins, 1972)

D_i/D_o	Nu_i	Nu_o
0	—	3.66
0.05	17.46	4.06
0.10	11.56	4.11
0.25	7.37	4.23
0.50	5.74	4.43
1.00	4.86	4.86

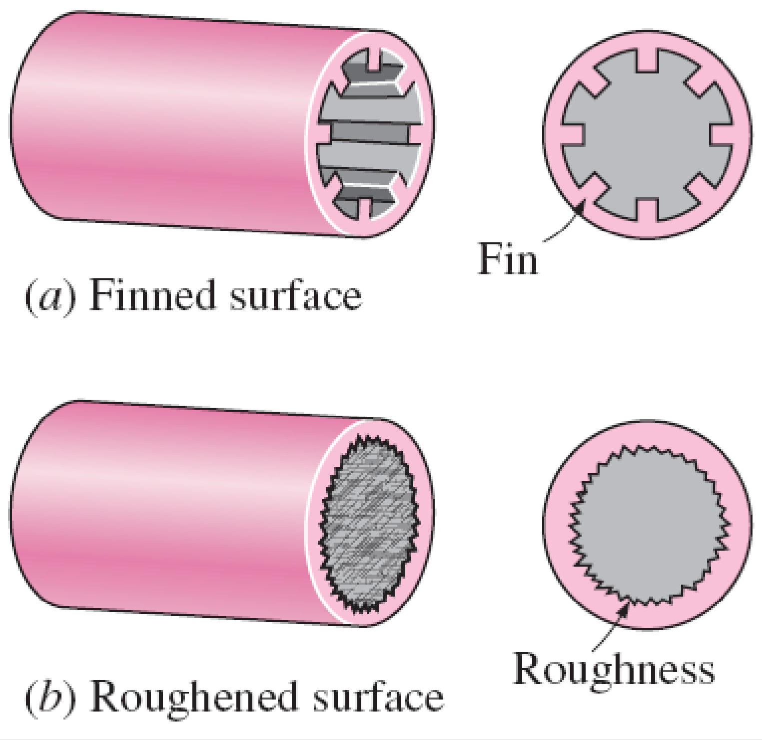
Heat Transfer Enhancement

Tubes with rough surfaces have much higher heat transfer coefficients than tubes with smooth surfaces.

Heat transfer in turbulent flow in a tube has been increased by as much as 400 percent by roughening the surface. Roughening the surface, of course, also increases the friction factor and thus the power requirement for the pump or the fan.

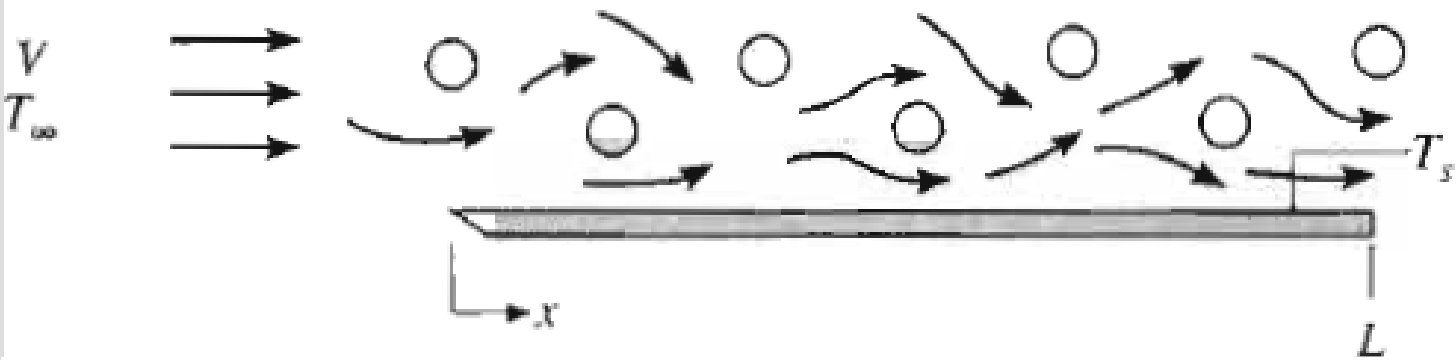
The convection heat transfer coefficient can also be increased by inducing pulsating flow by pulse generators, by inducing swirl by inserting a twisted tape into the tube, or by inducing secondary flows by coiling the tube.

Tube surfaces are often *roughened, corrugated,* or *finned* in order to *enhance* convection heat transfer.



Exercice:

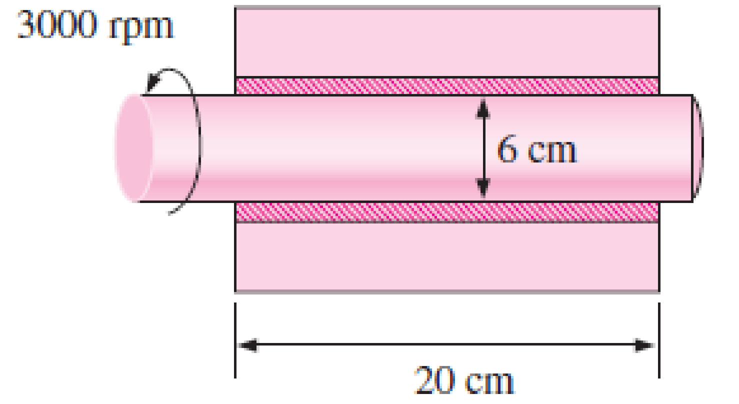
Parallel flow of atmospheric air over a flat plate of length $L = 3$ m is disrupted by an array of stationary rods placed in the flow path over the plate.



Laboratory measurements of the local convection coefficient at the surface of the plate are made for a prescribed value of V and $T_s > T_\infty$. The results are correlated by an expression of the form $h_x = 0.7 + 13.6x - 3.4x^2$, where h_x has units of $W/m^2 \cdot K$ and x is in meters. Evaluate the average convection coefficient $\bar{h}_L$ for the entire plate and the ratio $\bar{h}_L/h_L$ at the trailing edge.

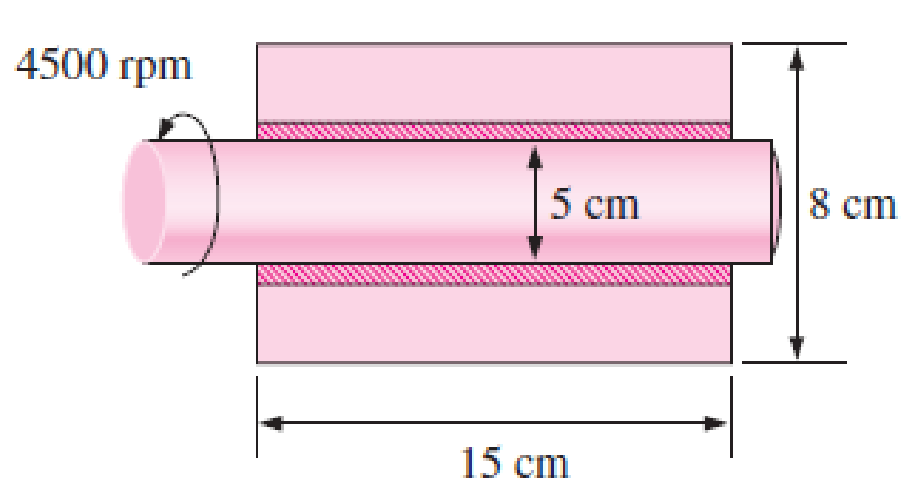
Exercice:

A 6-cm-diameter shaft rotates at 3000 rpm in a 20-cm-long bearing with a uniform clearance of 0.2 mm. At steady operating conditions, both the bearing and the shaft in the vicinity of the oil gap are at 50°C, and the viscosity and thermal conductivity of lubricating oil are 0.05 N · s/m^2 and 0.17 W/m · K. By simplifying and solving the continuity, momentum, and energy equations, determine (*a*) the maximum temperature of oil, (*b*) the rates of heat transfer to the bearing and the shaft, and (*c*) the mechanical power wasted by the viscous dissipation in the oil.



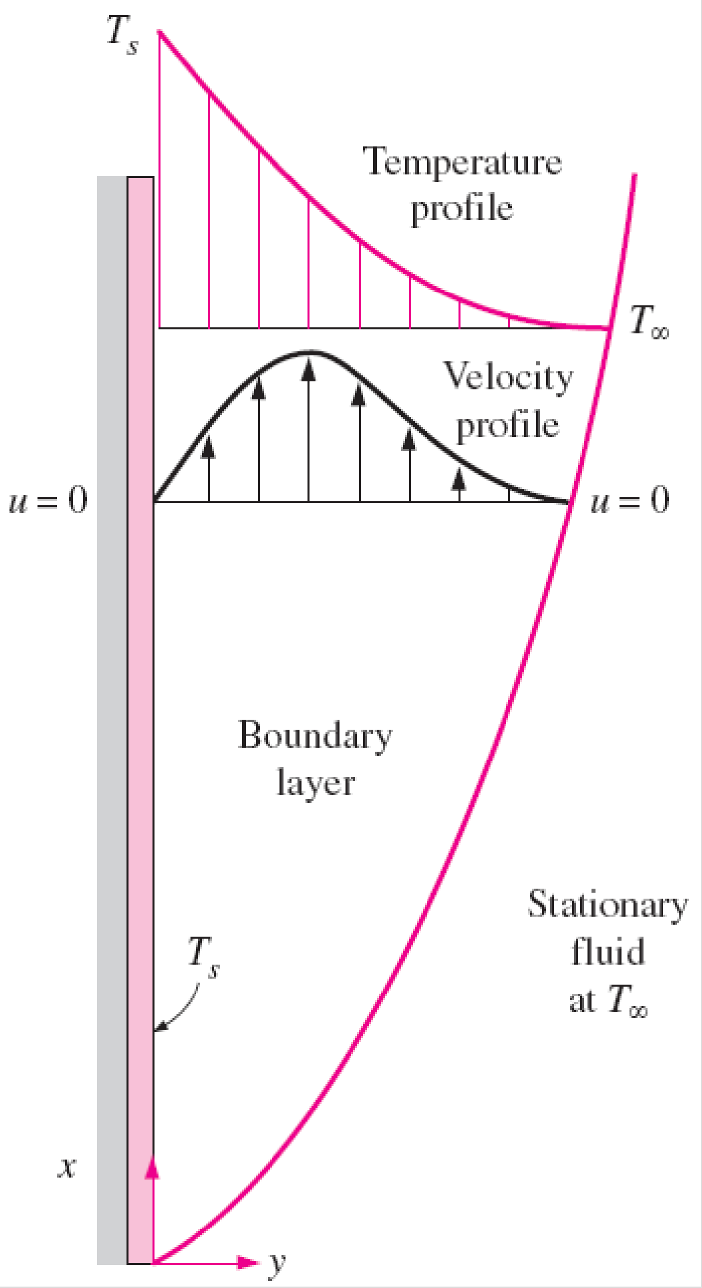
Quiz

A 5-cm-diameter shaft rotates at 4500 rpm in a 15-cm-long, 8-cm-outer-diameter cast iron bearing ($k = 70$ W/m · K) with a uniform clearance of 0.6 mm filled with lubricating oil ($\mu = 0.03$ N · s/m^2 and $k = 0.14$ W/m · K). The bearing is cooled externally by a liquid, and its outer surface is maintained at 40°C. Disregarding heat conduction through the shaft and assuming one-dimensional heat transfer, determine (*a*) the rate of heat transfer to the coolant, (*b*) the surface temperature of the shaft, and (*c*) the mechanical power wasted by the viscous dissipation in oil.



NATURAL CONVECTION

EQUATION OF MOTION AND THE GRASHOF NUMBER



The thickness of the boundary layer increases in the flow direction.

Unlike forced convection, the fluid velocity is *zero* at the outer edge of the velocity boundary layer as well as at the surface of the plate.

At the surface, the fluid temperature is equal to the plate temperature, and gradually decreases to the temperature of the surrounding fluid at a distance sufficiently far from the surface.

In the case of *cold surfaces,* the shape of the velocity and temperature profiles remains the same but their direction is reversed.

Typical velocity and temperature profiles for natural convection flow over a hot vertical plate at temperature T_s inserted in a fluid at temperature T_∞.

The Grashof Number

The governing equations of natural convection and the boundary conditions can be nondimensionalized by dividing all dependent and independent variables by suitable constant quantities:

$$x^* = \frac{x}{L_c} \qquad y^* = \frac{y}{L_c} \qquad u^* = \frac{u}{V} \qquad v^* = \frac{v}{V} \qquad \text{and} \qquad T^* = \frac{T - T_\infty}{T_s - T_\infty}$$

Substituting them into the momentum equation and simplifying give

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \left[\frac{g\beta(T_s - T_\infty)L_c^3}{\nu^2}\right] \frac{T^*}{\text{Re}_L^2} + \frac{1}{\text{Re}_L} \frac{\partial^2 u^*}{\partial {y^*}^2}$$

$$\text{Gr}_L = \frac{g\beta(T_s - T_\infty)L_c^3}{\nu^2}$$

Grashof number: Represents the natural convection effects in momentum equation

g = gravitational acceleration, m/s^2

β = coefficient of volume expansion, 1/K ($\beta = 1/T$ for ideal gases)

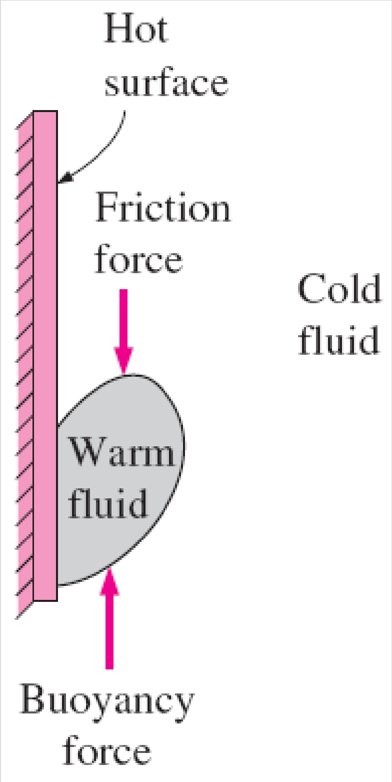
T_s = temperature of the surface, °C

T_∞ = temperature of the fluid sufficiently far from the surface, °C

L_c = characteristic length of the geometry, m

ν = kinematic viscosity of the fluid, m^2/s

- The Grashof number provides the main criterion in determining whether the fluid flow is laminar or turbulent in natural convection.
- For vertical plates, the critical Grashof number is observed to be about 10^9.



The Grashof number Gr is a measure of the relative magnitudes of the *buoyancy force* and the opposing *viscous force* acting on the fluid.

When a surface is subjected to external flow, the problem involves both natural and forced convection.

The relative importance of each mode of heat transfer is determined by the value of the coefficient Gr/Re^2:

- Natural convection effects are negligible if $Gr/Re^2 << 1$.
- Free convection dominates and the forced convection effects are negligible if $Gr/Re^2 >> 1$.
- Both effects are significant and must be considered if $Gr/Re^2 \approx 1$.