Fundamentals of Thermal-Fluid Sciences, 3rd Edition
Yunus A. Cengel, Robert H. Turner, John M. Cimbala
McGraw-Hill, 2008

HEAT TRANSFER FROM FINNED SURFACES

Mehmet Kanoglu

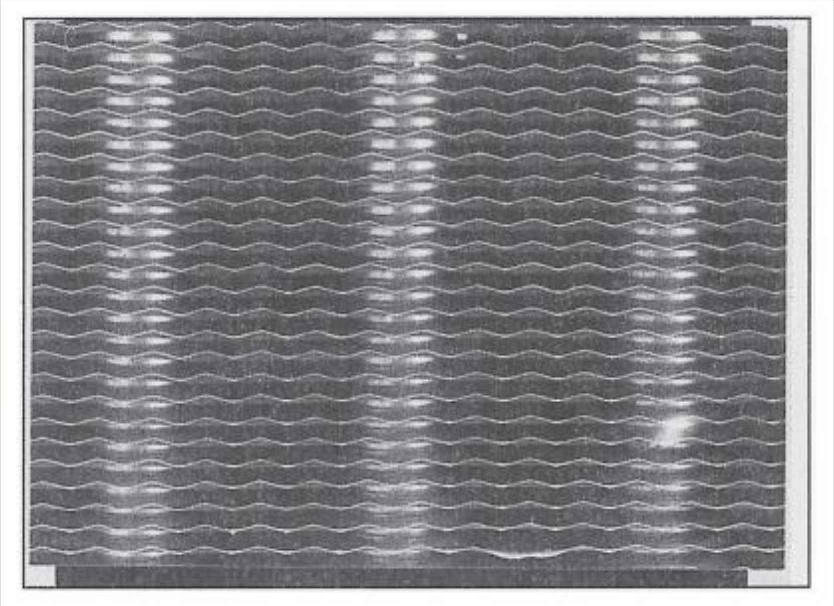
HEAT TRANSFER FROM FINNED SURFACES

$$\dot{Q}_{\text{conv}} = hA_s(T_s - T_\infty)$$

Newton's law of cooling: The rate of heat transfer from a surface to the surrounding medium

When T_s and T_∞ are fixed, *two ways* to increase the rate of heat transfer are

- To increase the *convection heat transfer coefficient h.* This may require the installation of a pump or fan, or replacing the existing one with a larger one, but this approach may or may not be practical. Besides, it may not be adequate.
- To increase the *surface area* A_s by attaching to the surface *extended surfaces* called ***fins*** made of highly conductive materials such as aluminum.



The thin plate fins of a car radiator greatly increase the rate of heat transfer to the air.

Some innovative fin designs.

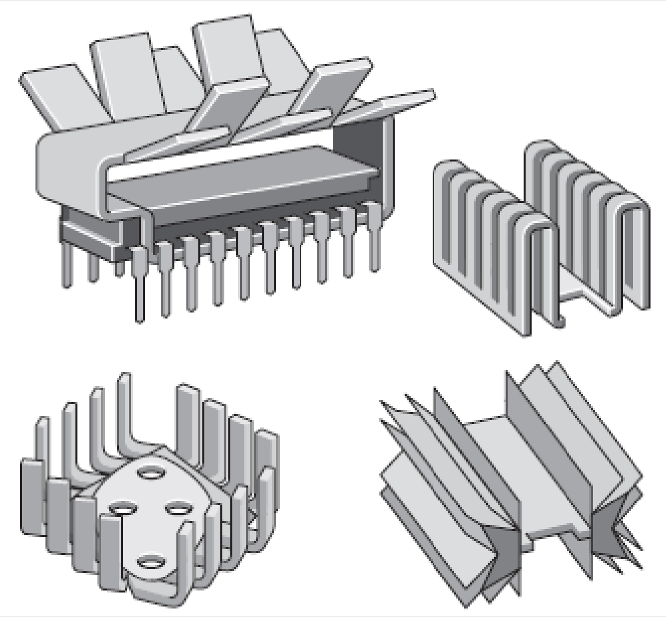


FIGURE 3–34

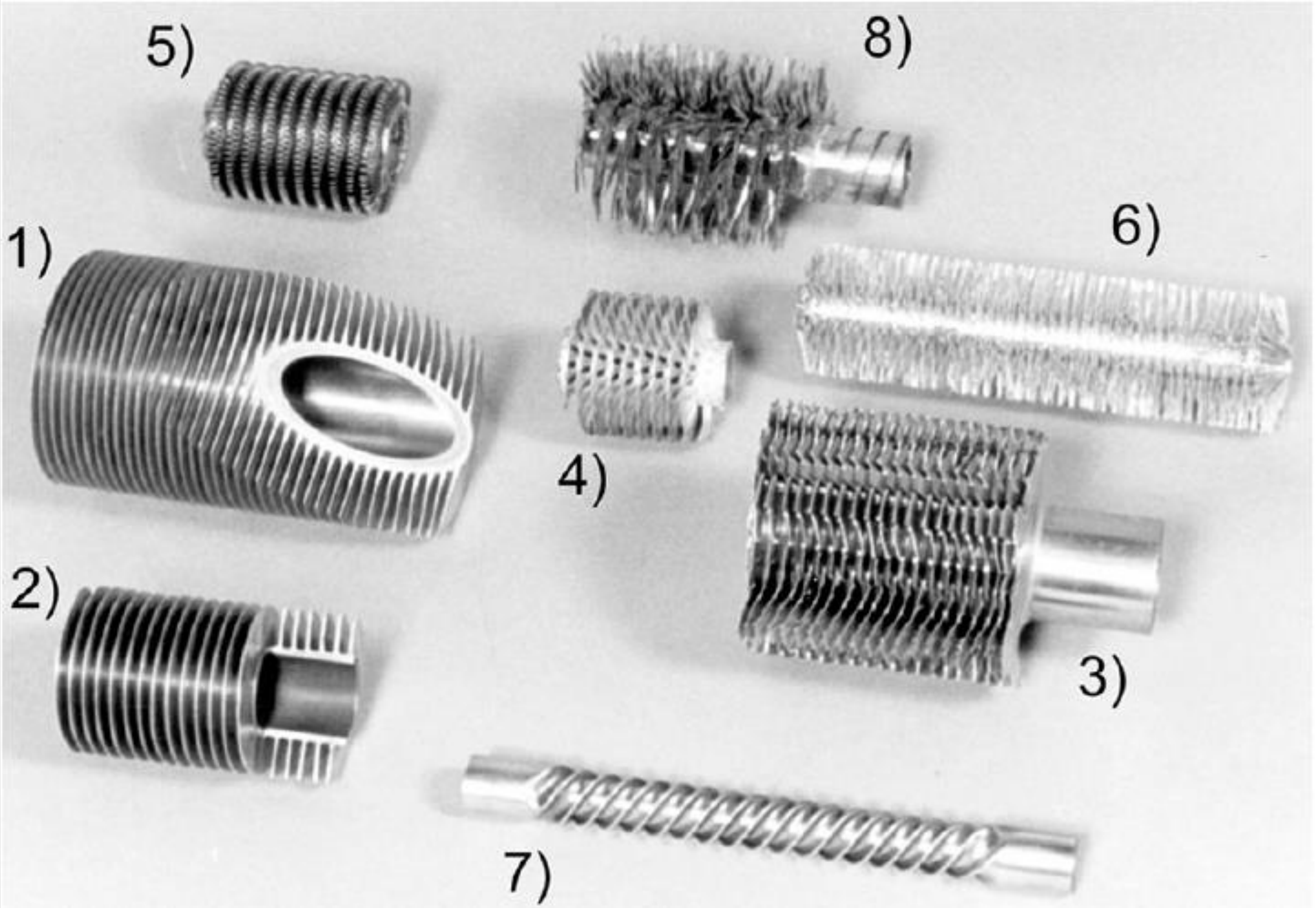
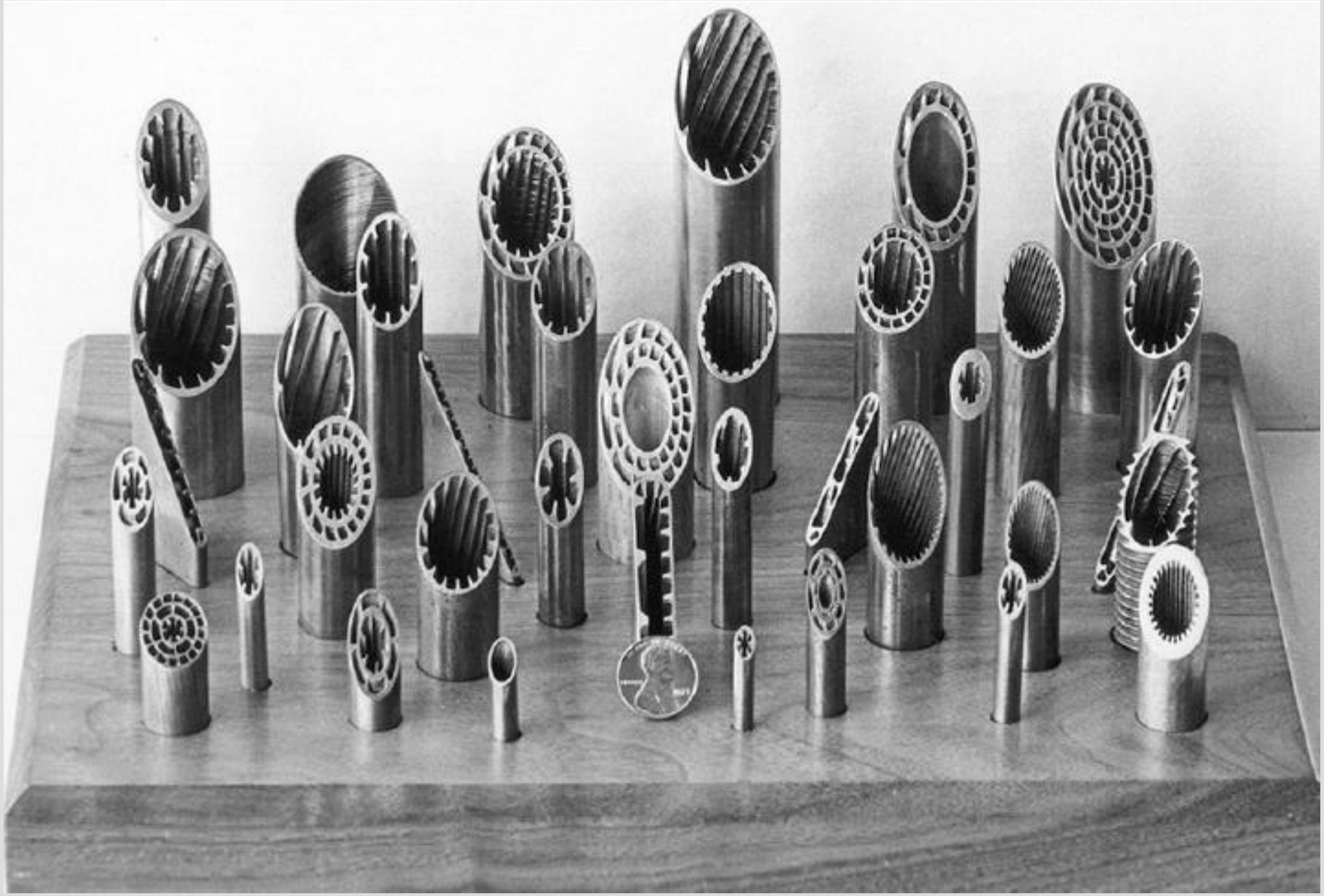


FIGURE 3–34



Performance Characteristics

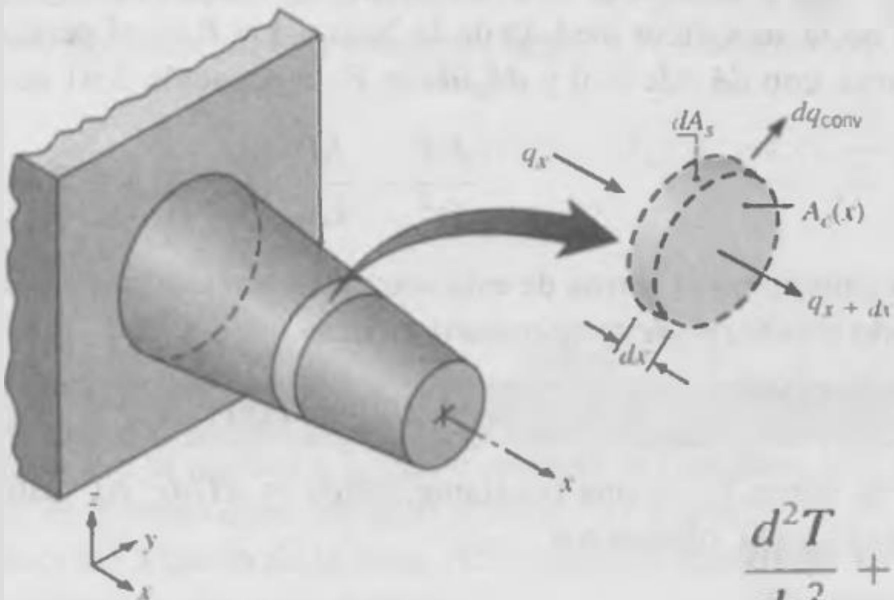
In this section we provide the performance characteristics:

- ❑ Temperature distribution,
- ❑ Rate of heat transfer,
- ❑ Fin efficiency

For convecting, radiating, and convecting-radiating fins. Configurations considered include longitudinal fins, radial fins, and spines.

Fin Equation

$$q_x = q_{x+dx} + dq_{conv}$$



$$\frac{d}{dx}\left(A_c \frac{dT}{dx}\right) - \frac{h}{k}\frac{dA_s}{dx}(T - T_\infty) = 0$$

o

$$\frac{d^2T}{dx^2} + \left(\frac{1}{A_c}\frac{dA_c}{dx}\right)\frac{dT}{dx} - \left(\frac{1}{A_c}\frac{h}{k}\frac{dA_s}{dx}\right)(T - T_\infty) = 0$$

$$q_x = -kA_c \frac{dT}{dx}$$

$$q_{x+dx} = q_x + \frac{dq_x}{dx} dx$$

$$dq_{conv} = h\, dA_s(T - T_\infty)$$

if A_c es una constante y $A_s = Px$.

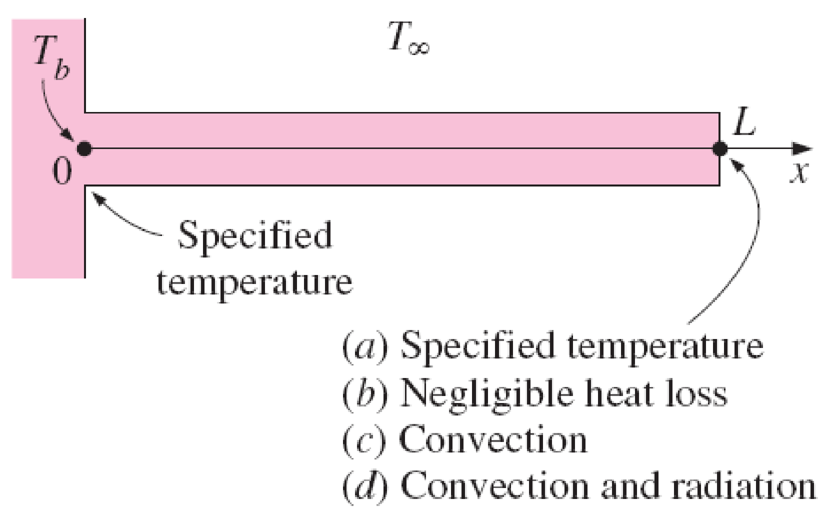
$$\frac{d^2\theta}{dx^2} - m^2\theta = 0$$

Differential equation

$$\theta = T - T_\infty$$

Temperature excess

$$m^2 = \frac{hp}{kA_c}$$



Boundary conditions at the fin base and the fin tip.

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0$$

$$\theta = T - T_\infty$$

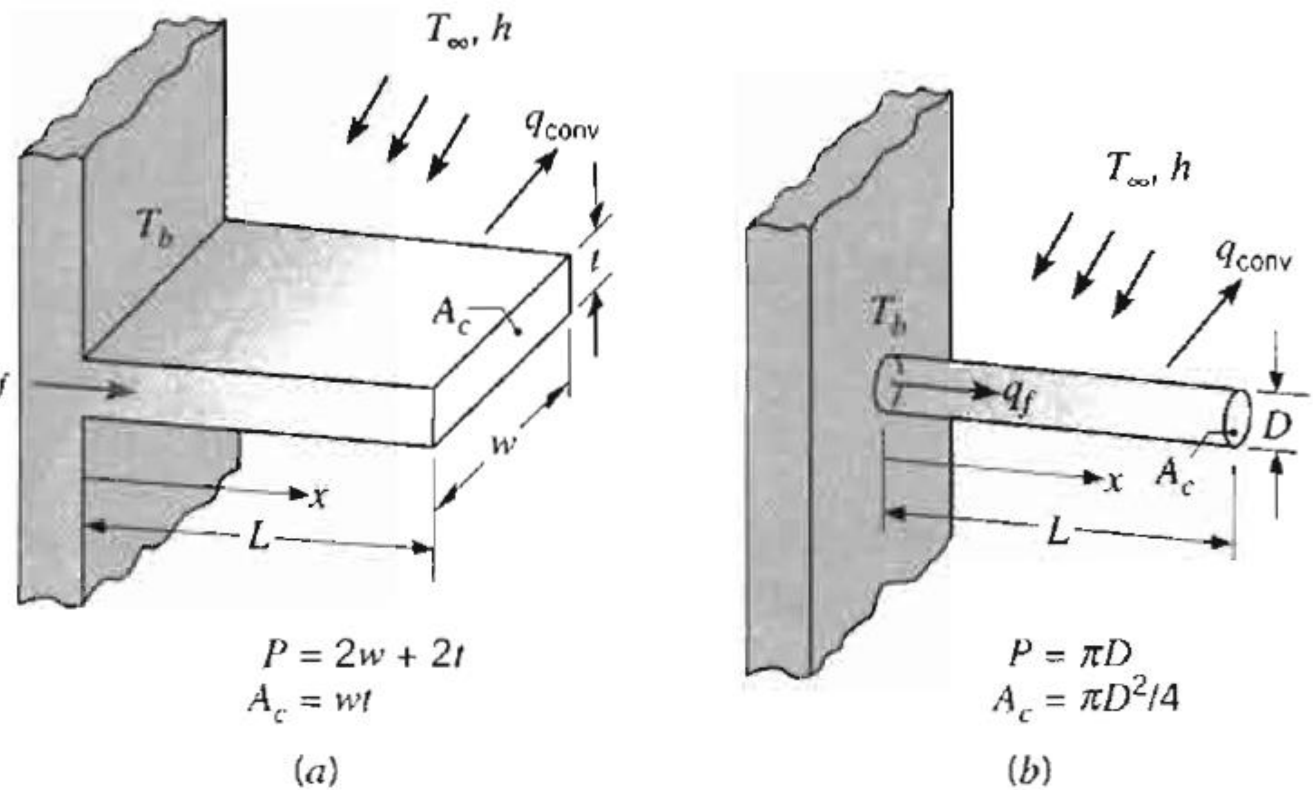
The general solution of the differential equation

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx}$$

$$m^2 \equiv \frac{hP}{kA_c}$$

$$\theta = T - T_\infty$$

$$\theta(0) = T_b - T_\infty \equiv \theta_b$$



Straight fins of uniform cross section. (*a*) Rectangular fin. (*b*) Pin fin.

Para evaluar las constantes C_1 y C_2 de la ecuación 3.66, es necesario especificar condiciones de frontera apropiadas. Una condición se especifica en términos de la temperatura en la *base* de la aleta ($x = 0$)

$$\theta(0) = T_b - T_\infty \equiv \theta_b$$

Constant base temperature and convecting tip

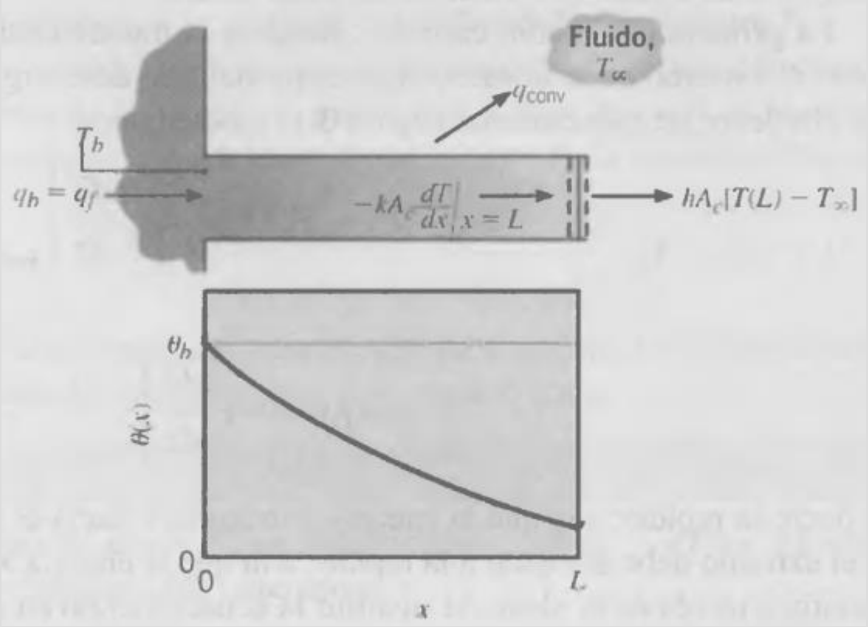
boundary conditions $h\theta(L) = -k\,d\theta/dx\big|_{x=L}$

$$\frac{\theta}{\theta_b} = \frac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL}$$

$$\dot{Q}_{\text{fin}} = \sqrt{hPkA_c}\,\theta_b \frac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}$$

$$\dot{Q}_{\text{fin,max}} = hA_{\text{fin}}\theta_b$$

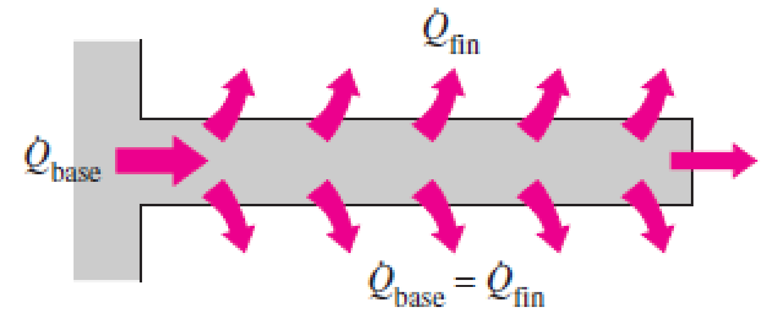
$$\eta_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin, max}}}$$



Constant base temperature and insulated tip

Fins are not likely to be so long that their temperature approaches the surrounding temperature at the tip. A more realistic assumption is for heat transfer from the fin tip to be negligible since the surface area of the fin tip is usually a negligible fraction of the total fin area.

boundary conditions $d\theta / dx\big|_{x=L} = 0$



$$\frac{\theta}{\theta_b} = \frac{\cosh m(L-x)}{\cosh mL}$$

$$\dot{Q}_{\text{fin}} = \sqrt{hPkA_c}\,\theta_b \tanh mL \qquad \dot{Q}_{\text{fin,max}} = hA_{\text{fin}}\,\theta_b$$

$$\eta_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin, max}}} = \frac{\tanh mL}{mL}$$

Convection (or Combined Convection and Radiation) from Fin Tip

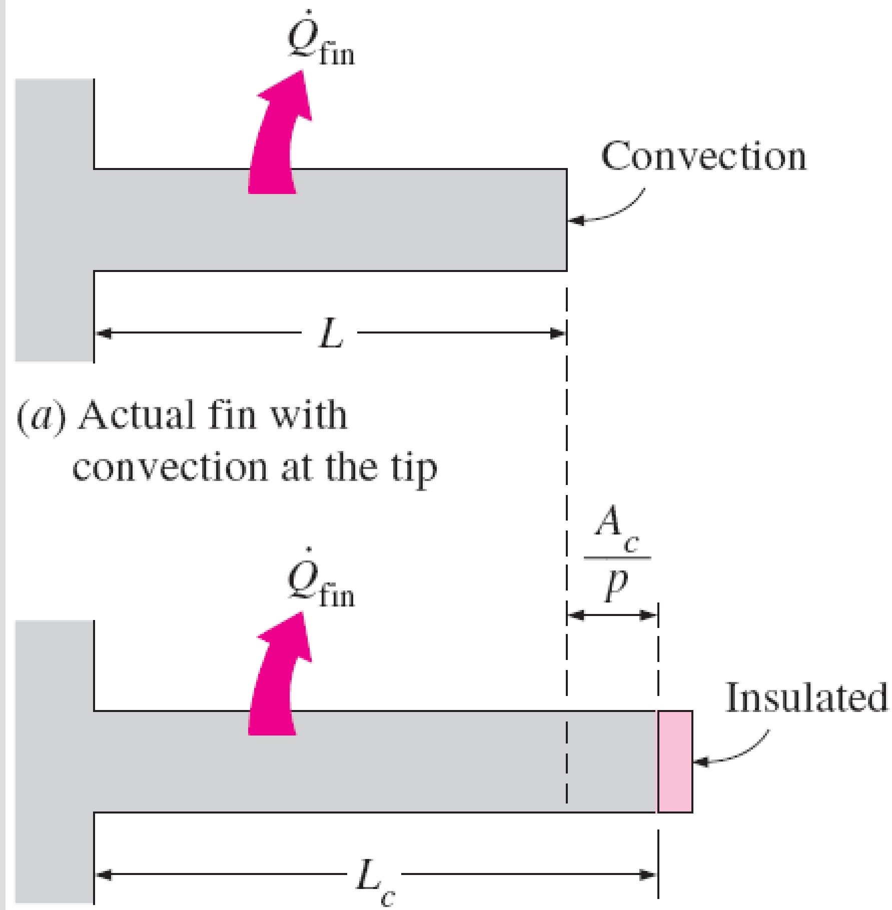
A practical way of accounting for the heat loss from the fin tip is to replace the *fin length L* in the relation for the *insulated tip* case by a **corrected length** defined as

$$L_c = L + \frac{A_c}{p}$$

$$L_{c,\text{ rectangular fin}} = L + \frac{t}{2}$$

$$L_{c,\text{ cylindrical fin}} = L + \frac{D}{4}$$

t the thickness of the rectangular fins
D the diameter of the cylindrical fins



(*a*) Actual fin with convection at the tip

(*b*) Equivalent fin with insulated tip

Corrected fin length L_c is defined such that heat transfer from a fin of length L_c with insulated tip is equal to heat transfer from the actual fin of length L with convection at the fin tip.

Constant base and tip temperatures

boundary conditions $\theta(\mathrm{L}) = \theta_{\mathrm{L}}$

$$\frac{\theta}{\theta_b} = \frac{(\theta_t / \theta_b)\sinh mx + \sinh m(L - x)}{\sinh mL}$$

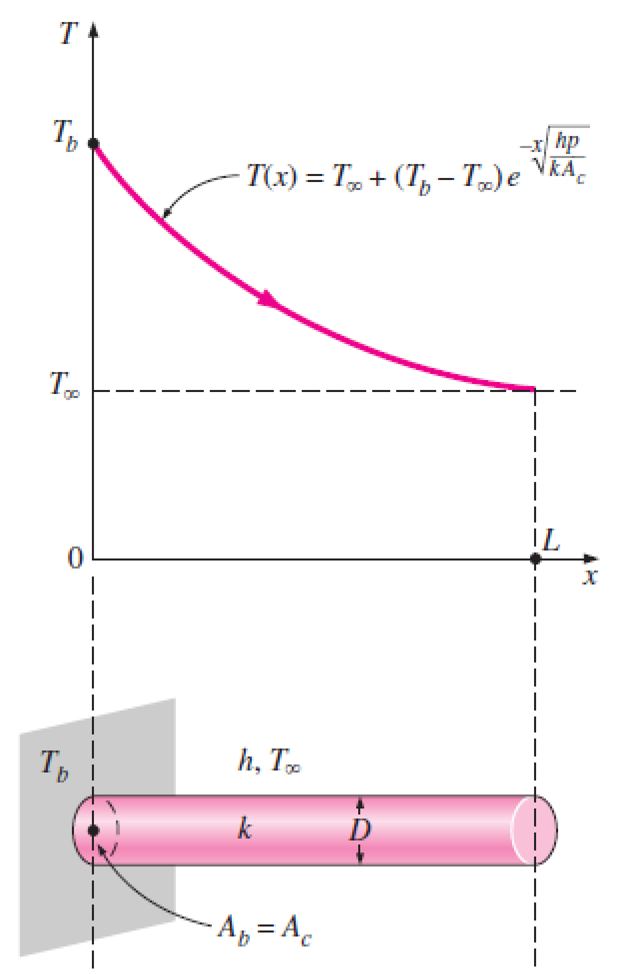
$$\dot{Q}_{\text{fin}} = \sqrt{hPkA_c}\,\theta_b \frac{\cosh mL - (\theta_t / \theta_b)}{\sinh mL}$$

$$\dot{Q}_{\text{fin,max}} = hA_{\text{fin}}\theta_b$$

$$\eta_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin, max}}}$$

Infinitely high fin with constant base temperature

$$\text{boundary conditions}:(\mathrm{L} \to \infty) \qquad \theta(\mathrm{L}) = 0$$



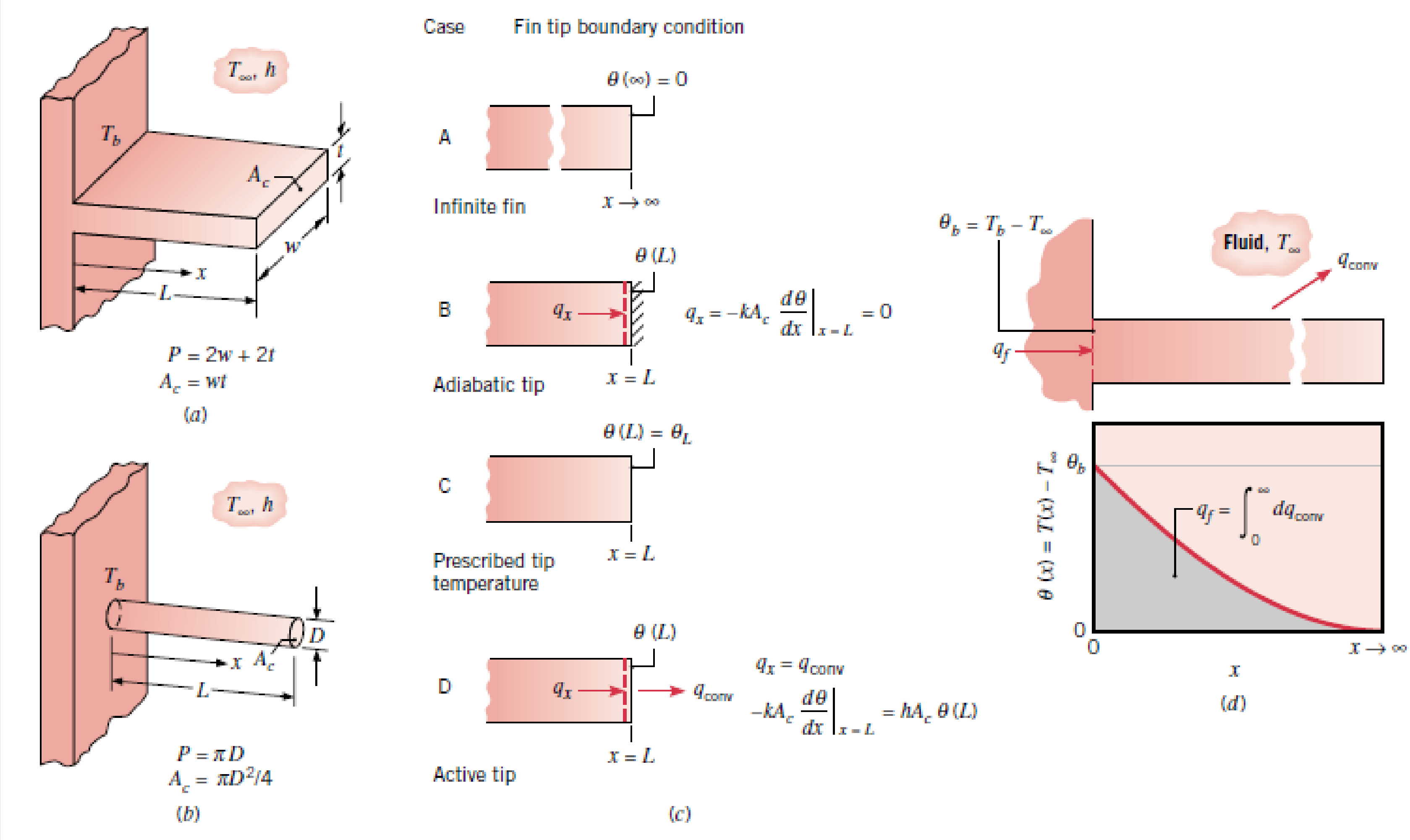
$$\frac{\theta}{\theta_b} = e^{-mx}$$

$$\dot{Q}_{\text{fin}} = \sqrt{hPkA_c}\,\theta_b$$

$$\dot{Q}_{\text{fin,max}} = hA_{\text{fin}}\,\theta_b$$

$$\eta_{fin} = \sqrt{\frac{kP}{hA_c}}$$

Revisar eficiencia!



Conduction and convection in a straight fin of uniform cross-sectional area. (a) Rectangular fin. (b) Pin fin. (c) Four common tip boundary conditions. (d) Temperature distribution for the infinite fin ($x \to \infty$)

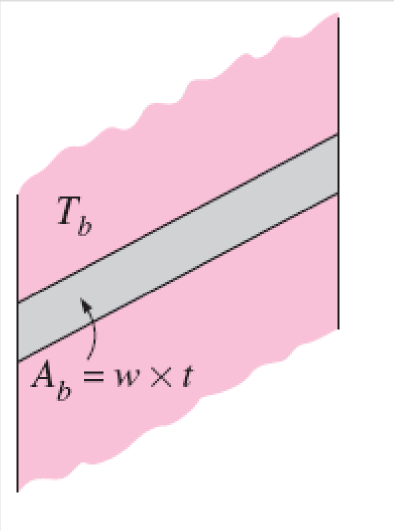
Temperature distribution, and loss heat of uniform sectional fins

Caso	Condición de aleta $(x = L)$	Distribución de temperaturas θ/θ_b	Transferencia de calor de la aleta q_f
A	Transferencia de calor por convección: $h\theta(L) = -kd\theta/dx\|_{x=L}$	$\dfrac{\cosh m(L-x) + (h/mk)\,\operatorname{senh} m(L-x)}{\cosh mL + (h/mk)\,\operatorname{senh} mL}$ (3.70)	$M\dfrac{\operatorname{senh} mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\operatorname{senh} mL}$ (3.72)
B	Adiabática: $d\theta/dx\|_{x=L} = 0$	$\dfrac{\cosh m(L-x)}{\cosh mL}$ (3.75)	$M \tanh mL$ (3.76)
C	Temperatura establecida: $\theta(L) = \theta_L$	$\dfrac{(\theta_L/\theta_b)\operatorname{senh} mx + \operatorname{senh} m(L-x)}{\operatorname{senh} mL}$ (3.77)	$M\dfrac{(\cosh mL - \theta_L/\theta_b)}{\operatorname{senh} mL}$ (3.78)
D	Aleta infinita $(L \rightarrow \infty)$: $\theta(L) = 0$	e^{-mx} (3.79)	M (3.80)

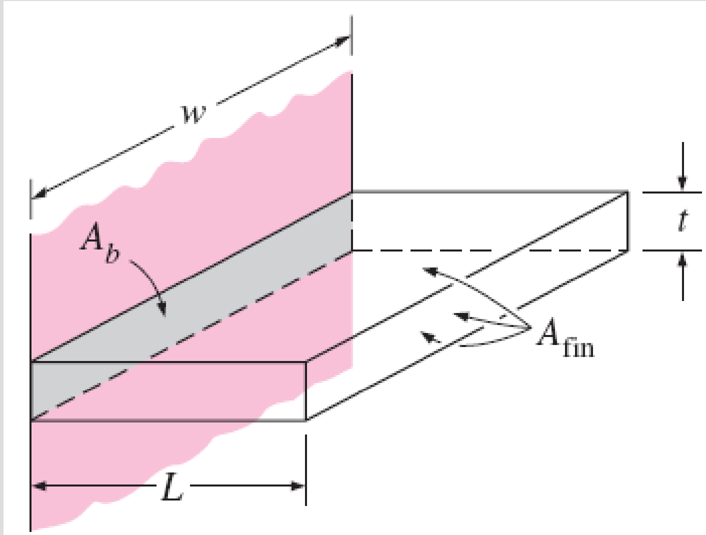
$\theta \equiv T - T_\infty$ $\qquad m^2 \equiv hP/kA_c$

$\theta_b = \theta(0) = T_b - T_\infty$ $\qquad M \equiv \sqrt{hPkA_c}\,\theta_b$

Fin Efficiency



(*a*) Surface without fins

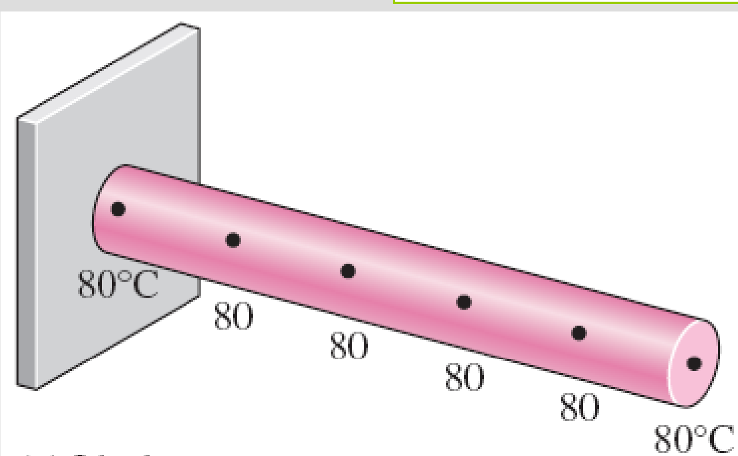


(*b*) Surface with a fin

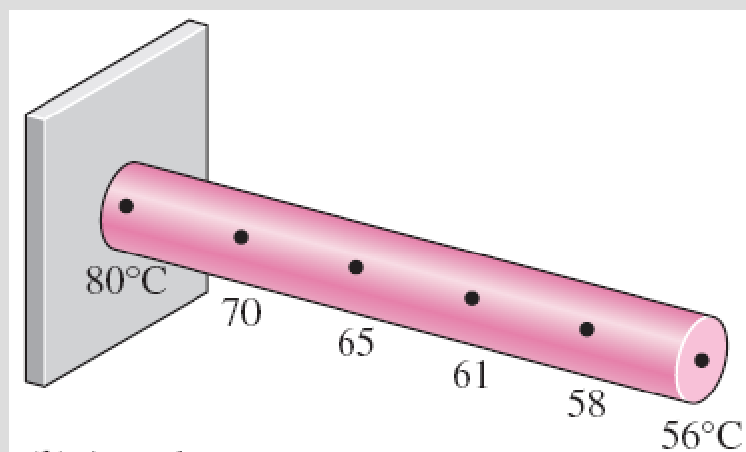
$$A_{\text{fin}} = 2 \times w \times L + w \times t$$
$$\cong 2 \times w \times L$$

$$\dot{Q}_{\text{fin, max}} = hA_{\text{fin}}\,(T_b - T_\infty)$$

Fins enhance heat transfer from a surface by enhancing surface area.



(*a*) Ideal



(*b*) Actual

Ideal and actual temperature distribution along a fin.

$$\dot{Q}_{\text{fin, max}} = hA_{\text{fin}}(T_b - T_\infty)$$

Zero thermal resistance or *infinite thermal conductivity* ($T_{\text{fin}} = T_b$)

$$\eta_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin, max}}} = \frac{\text{Actual heat transfer rate from the fin}}{\begin{array}{c}\text{Ideal heat transfer rate from the fin}\\ \text{if the entire fin were at base temperature}\end{array}}$$

$$\dot{Q}_{\text{fin}} = \eta_{\text{fin}}\,\dot{Q}_{\text{fin, max}} = \eta_{\text{fin}}\,hA_{\text{fin}}(T_b - T_\infty)$$

$$\eta_{\text{long fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin, max}}} = \frac{\sqrt{hpkA_c}\,(T_b - T_\infty)}{hA_{\text{fin}}(T_b - T_\infty)} = \frac{1}{L}\sqrt{\frac{kA_c}{hp}} = \frac{1}{mL}$$

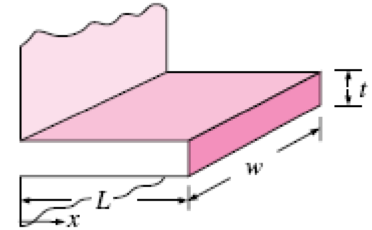
$$\eta_{\text{adiabatic tip}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin, max}}} = \frac{\sqrt{hpkA_c}\,(T_b - T_\infty)\tanh aL}{hA_{\text{fin}}(T_b - T_\infty)} = \frac{\tanh mL}{mL}$$

Efficiency and surface areas of common fin configurations

Straight rectangular fins

$m = \sqrt{2h/kt}$
$L_c = L + t/2$
$A_{\text{fin}} = 2wL_c$

$$\eta_{\text{fin}} = \frac{\tanh mL_c}{mL_c}$$

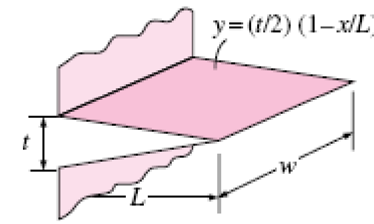


Straight triangular fins

$m = \sqrt{2h/kt}$

$A_{\text{fin}} = 2w\sqrt{L^2 + (t/2)^2}$

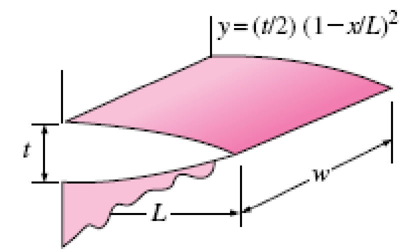
$$\eta_{\text{fin}} = \frac{1}{mL}\frac{I_1(2mL)}{I_0(2mL)}$$



Straight parabolic fins

$m = \sqrt{2h/kt}$
$A_{\text{fin}} = wL[C_1 + (L/t)\ln(t/L + C_1)]$
$C_1 = \sqrt{1 + (t/L)^2}$

$$\eta_{\text{fin}} = \frac{2}{1 + \sqrt{(2mL)^2 + 1}}$$

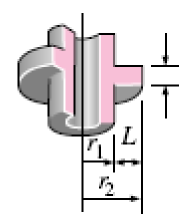


Circular fins of rectangular profile

$m = \sqrt{2h/kt}$
$r_{2c} = r_2 + t/2$
$A_{fin} = 2\pi(r_{2c}^2 - r_1^2)$

$$\eta_{\text{fin}} = C_2\frac{K_1(mr_1)I_1(mr_{2c}) - I_1(mr_1)K_1(mr_{2c})}{I_0(mr_1)K_1(mr_{2c}) + K_0(mr_1)I_1(mr_{2c})}$$

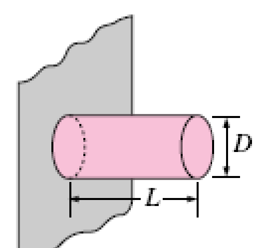
$$C_2 = \frac{2r_1/m}{r_{2c}^2 - r_1^2}$$



Pin fins of rectangular profile

$m = \sqrt{4h/kD}$
$L_c = L + D/4$
$A_{\text{fin}} = \pi DL_c$

$$\eta_{\text{fin}} = \frac{\tanh mL_c}{mL_c}$$

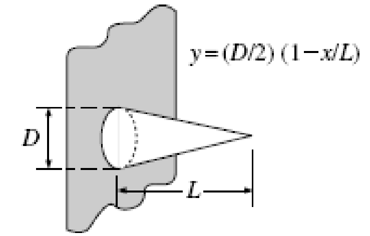


Pin fins of triangular profile

$$m = \sqrt{4h/kD}$$

$$A_{\text{fin}} = \frac{\pi D}{2}\sqrt{L^2 + (D/2)^2}$$

$$\eta_{\text{fin}} = \frac{2}{mL}\frac{I_2(2mL)}{I_1(2mL)}$$



Pin fins of parabolic profile

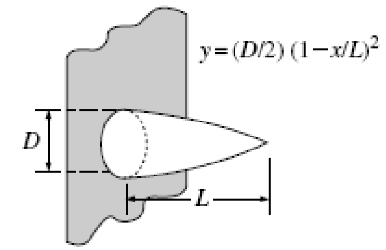
$$m = \sqrt{4h/kD}$$

$$A_{\text{fin}} = \frac{\pi L^3}{8D}\left[C_3 C_4 - \frac{L}{2D} ln(2DC_4/L + C_3)\right]$$

$$C_3 = 1 + 2(D/L)^2$$

$$C_4 = \sqrt{1 + (D/L)^2}$$

$$\eta_{\text{fin}} = \frac{2}{1 + \sqrt{(2mL/3)^2 + 1}}$$

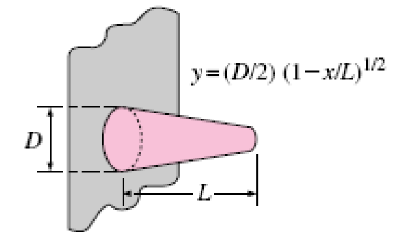


Pin fins of parabolic profile (blunt tip)

$$m = \sqrt{4h/kD}$$

$$A_{\text{fin}} = \frac{\pi D^4}{96L^2}\left\{[16(L/D)^2 + 1]^{3/2} - 1\right\}$$

$$\eta_{\text{fin}} = \frac{3}{2mL}\frac{I_1(4mL/3)}{I_0(4mL/3)}$$



- Fins with triangular and parabolic profiles contain less material and are more efficient than the ones with rectangular profiles.
- The fin efficiency decreases with increasing fin length. Why?
- How to choose fin length? Increasing the length of the fin beyond a certain value cannot be justified unless the added benefits outweigh the added cost.
- Fin lengths that cause the fin efficiency to drop below 60 percent usually cannot be justified economically.
- The efficiency of most fins used in practice is above 90 percent.

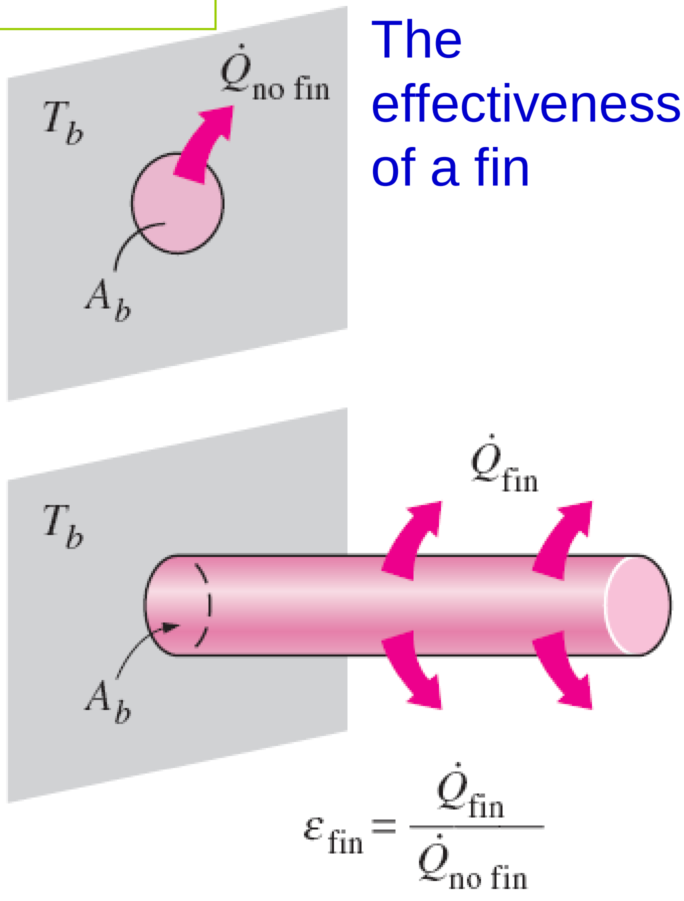
Fin Effectiveness

$$\varepsilon_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{no fin}}} = \frac{\dot{Q}_{\text{fin}}}{hA_b\,(T_b - T_\infty)} = \frac{\text{Heat transfer rate from the fin of } \textit{base area } A_b}{\text{Heat transfer rate from the surface of } \textit{area } A_b}$$

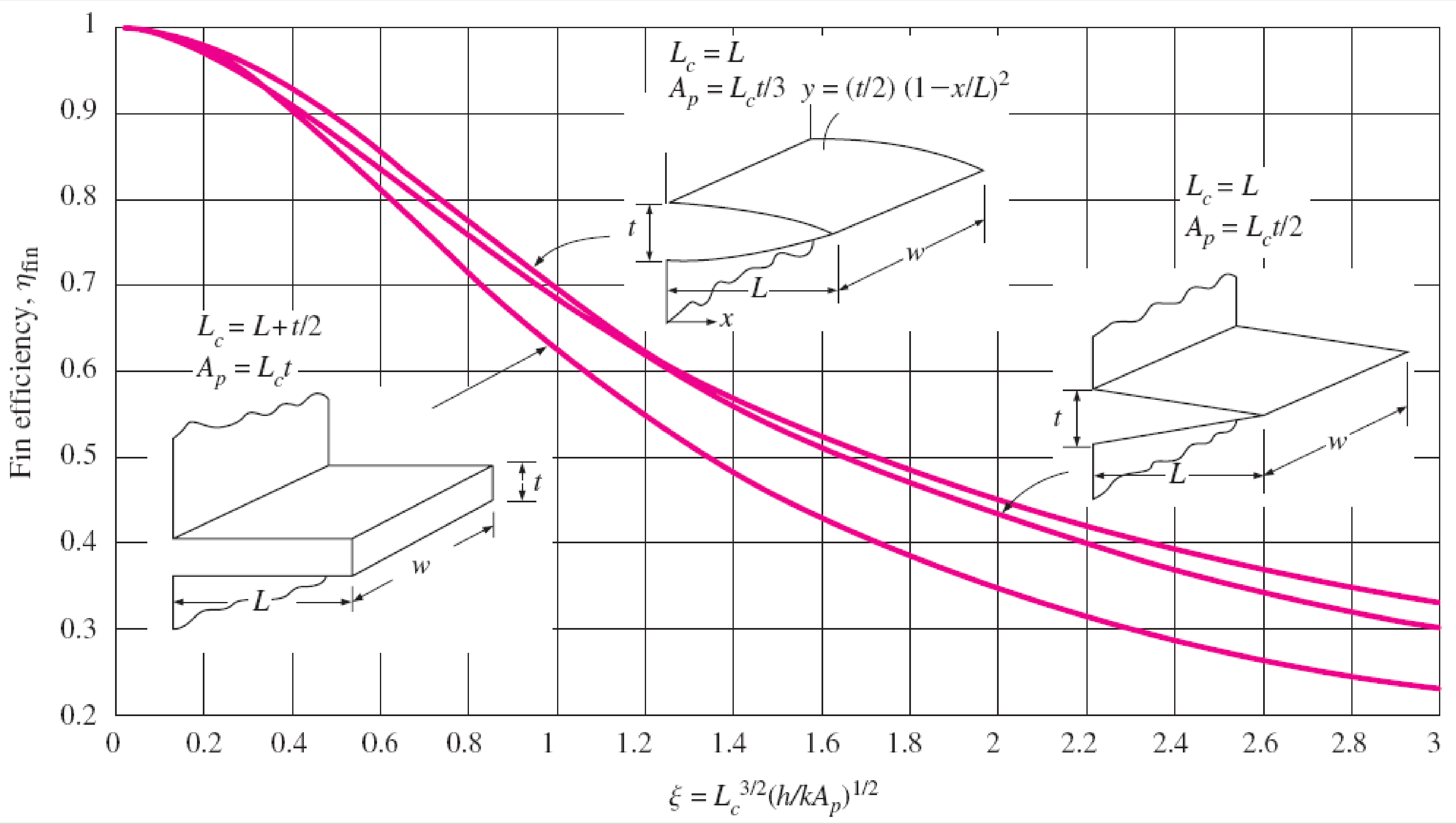
$$\varepsilon_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{no fin}}} = \frac{\dot{Q}_{\text{fin}}}{hA_b\,(T_b - T_\infty)} = \frac{\eta_{\text{fin}}\, hA_{\text{fin}}\,(T_b - T_\infty)}{hA_b\,(T_b - T_\infty)} = \frac{A_{\text{fin}}}{A_b}\,\eta_{\text{fin}}$$

$$\varepsilon_{\text{long fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{no fin}}} = \frac{\sqrt{hpkA_c}\,(T_b - T_\infty)}{hA_b\,(T_b - T_\infty)} = \sqrt{\frac{kp}{hA_c}}$$

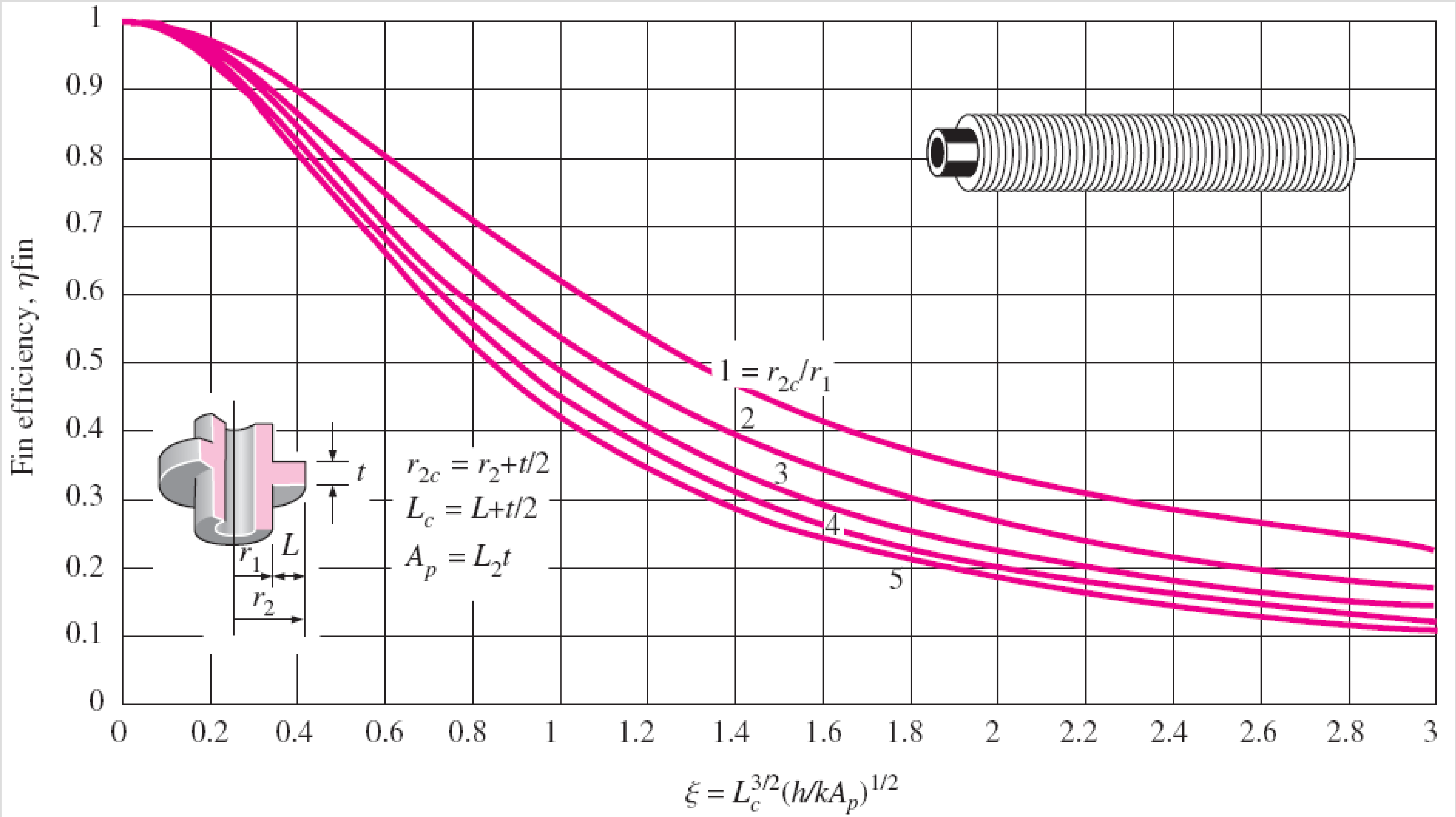
- The *thermal conductivity* ***k*** of the fin should be as high as possible. Use aluminum, copper, iron.
- The ratio of the *perimeter* to the *cross-sectional area* of the fin ***p*/*A*$_c$** should be as high as possible. Use slender pin fins.
- *Low convection heat transfer coefficient* ***h****.* Place fins on gas (air) side.
- The use of fins are recommended when $\varepsilon_f \geq 2$. (Incropera)



The effectiveness of a fin



Efficiency of straight fins of rectangular, triangular, and parabolic profiles.



Efficiency of annular fins of constant thickness *t*.

The total rate of heat transfer from a finned surface

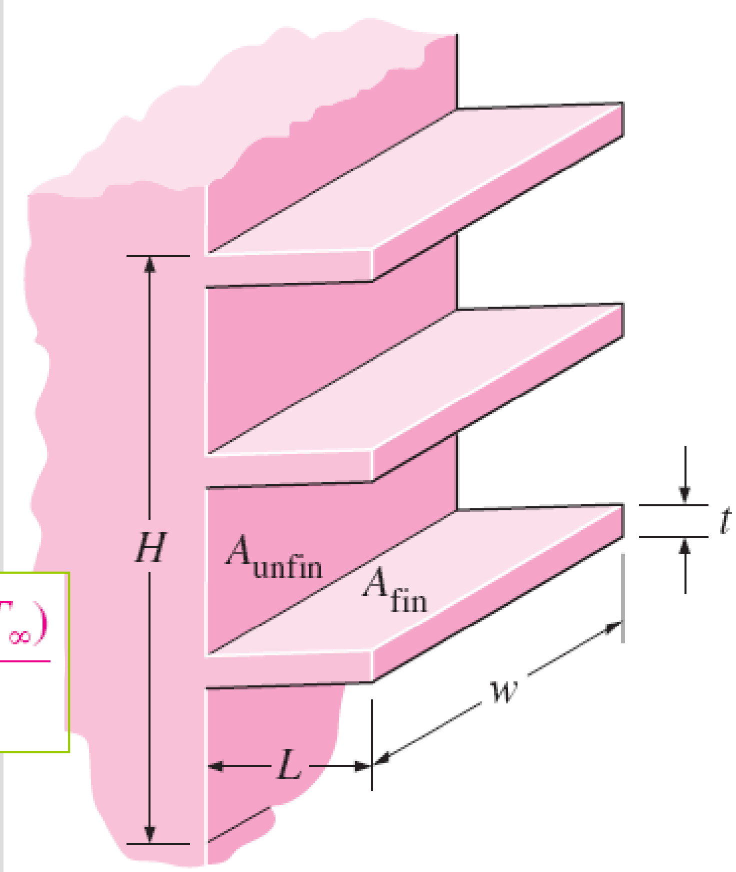
$$
\begin{aligned}
\dot{Q}_{\text{total, fin}} &= \dot{Q}_{\text{unfin}} + \dot{Q}_{\text{fin}} \\
&= hA_{\text{unfin}}(T_b - T_\infty) + \eta_{\text{fin}} hA_{\text{fin}}(T_b - T_\infty) \\
&= h(A_{\text{unfin}} + \eta_{\text{fin}} A_{\text{fin}})(T_b - T_\infty)
\end{aligned}
$$

Overall effectiveness for a finned surface

$$
\varepsilon_{\text{fin, overall}} = \frac{\dot{Q}_{\text{total, fin}}}{\dot{Q}_{\text{total, no fin}}} = \frac{h(A_{\text{unfin}} + \eta_{\text{fin}} A_{\text{fin}})(T_b - T_\infty)}{hA_{\text{no fin}}(T_b - T_\infty)}
$$

The overall fin effectiveness depends on the fin density (number of fins per unit length) as well as the effectiveness of the individual fins.

The overall effectiveness is a better measure of the performance of a finned surface than the effectiveness of the individual fins.

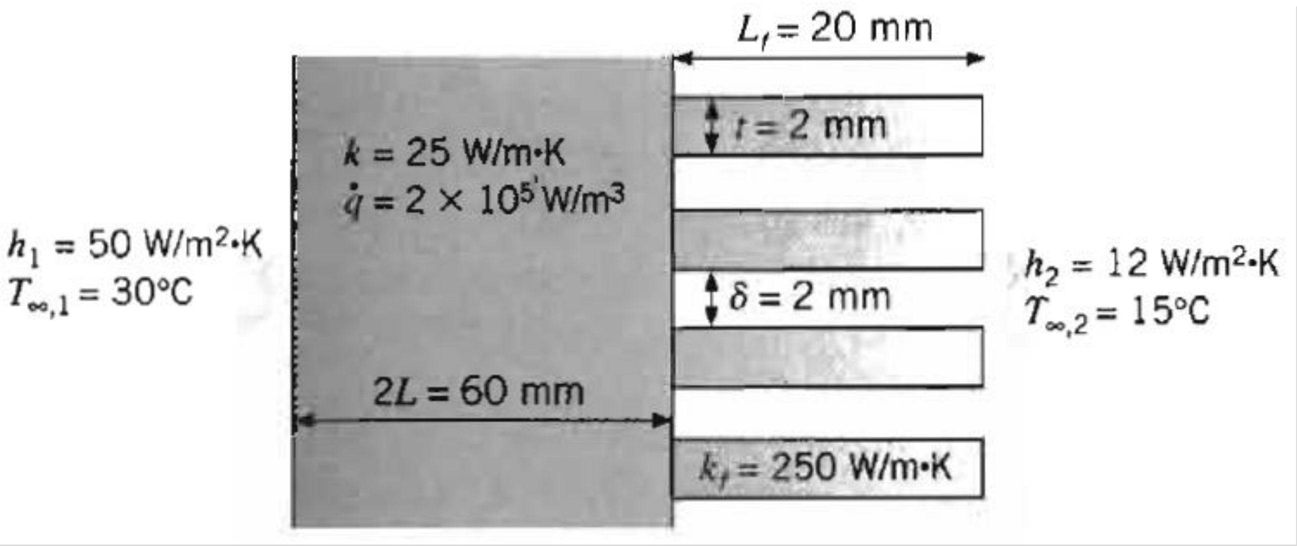


$$
\begin{aligned}
A_{\text{no fin}} &= w \times H \\
A_{\text{unfin}} &= w \times H - 3 \times (t \times w) \\
A_{\text{fin}} &= 2 \times L \times w + t \times w \\
&\cong 2 \times L \times w \text{ (one fin)}
\end{aligned}
$$

Various surface areas associated with a rectangular surface with three fins.

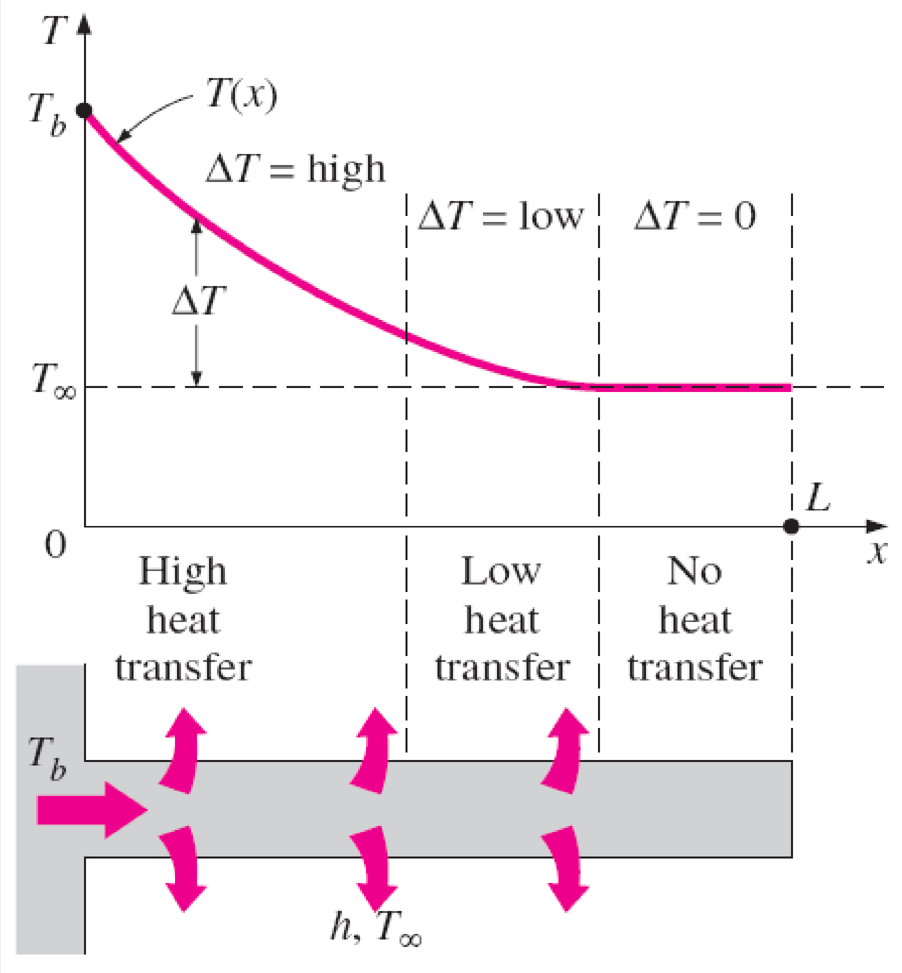
Work Class:

Heat is uniformly generated at the rate of 2×10^5 W/m^3 in a wall of thermal conductivity 25 W/m · K and thickness 60 mm. The wall is exposed to convection on both sides, with different heat transfer coefficients and temperatures as shown. There are straight rectangular fins on the right-hand side of the wall, with dimensions as shown and thermal conductivity of 250 W/m · K. What is the maximum temperature that will occur in the wall?



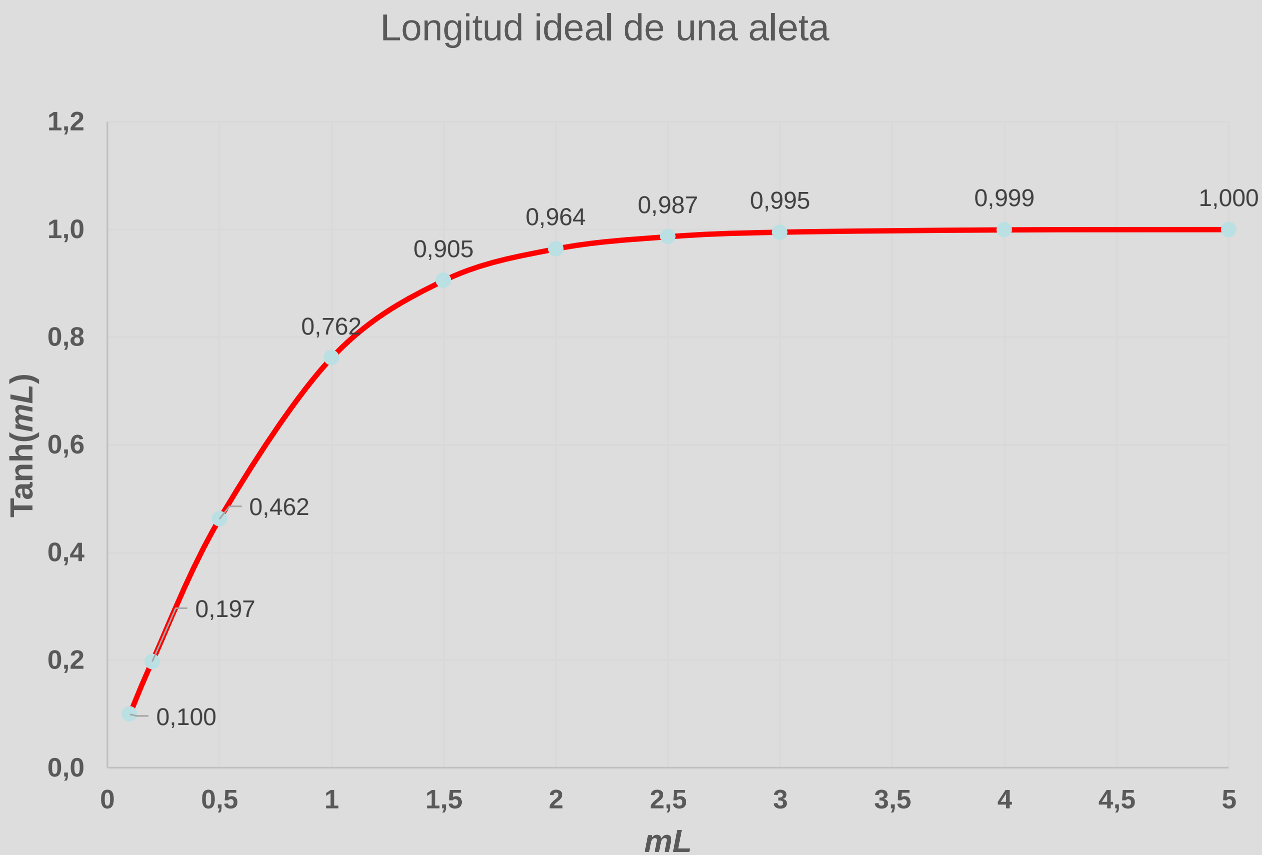
Proper Length of a Fin

$$\frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{long fin}}} = \frac{\sqrt{hpkA_c}\,(T_b - T_\infty)\tanh mL}{\sqrt{hpkA_c}\,(T_b - T_\infty)} = \tanh mL$$



Because of the gradual temperature drop along the fin, the region near the fin tip makes little or no contribution to heat transfer.

Proper Length of a Fin



The variation of heat transfer from a fin relative to that from an infinitely long fin

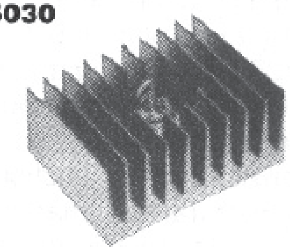
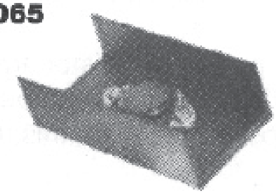
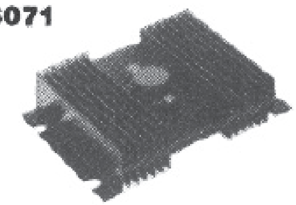
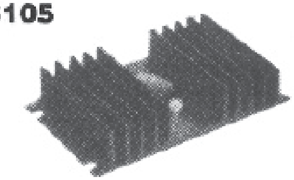
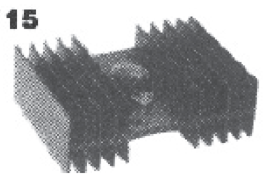
mL	$\dfrac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{long fin}}} = \tanh mL$
0.1	0.100
0.2	0.197
0.5	0.462
1.0	0.762
1.5	0.905
2.0	0.964
2.5	0.987
3.0	0.995
4.0	0.999
5.0	1.000

mL = 5 → an infinitely long fin *mL* = 1 offer a good compromise between heat transfer performance and the fin size.

- **Heat sinks:** Specially designed finned surfaces which are commonly used in the cooling of electronic equipment, and involve one-of-a-kind complex geometries.
- The heat transfer performance of heat sinks is usually expressed in terms of their *thermal resistances R.*
- A small value of thermal resistance indicates a small temperature drop across the heat sink, and thus a high fin efficiency.

$$\boxed{\dot{Q}_{\text{fin}} = \frac{T_b - T_\infty}{R} = hA_{\text{fin}}\,\eta_{\text{fin}}\,(T_b - T_\infty)}$$

Combined natural convection and radiation thermal resistance of various heat sinks used in the cooling of electronic devices between the heat sink and the surroundings. All fins are made of aluminum 6063T-5, are black anodized, and are 76 mm (3 in) long.

HS 5030		R = 0.9°C/W (vertical) R = 1.2°C/W (horizontal) Dimensions: 76 mm × 105 mm × 44 mm Surface area: 677 cm^2
HS 6065		R = 5°C/W Dimensions: 76 mm × 38 mm × 24 mm Surface area: 387 cm^2
HS 6071		R = 1.4°C/W (vertical) R = 1.8°C/W (horizontal) Dimensions: 76 mm × 92 mm × 26 mm Surface area: 968 cm^2
HS 6105		R = 1.8°C/W (vertical) R = 2.1°C/W (horizontal) Dimensions: 76 mm × 127 mm × 91 mm Surface area: 677 cm^2
HS 6115		R = 1.1°C/W (vertical) R = 1.3°C/W (horizontal) Dimensions: 76 mm × 102 mm × 25 mm Surface area: 929 cm^2

Fin Design

The measures η_f and ε_f probably attract the interest of designers not because their absolute values guide the designs, but because they are useful in characterizing fins with more complex shapes. In such cases the solutions are often so complex that η_f and ε_f plots serve as labor saving graphical solutions.

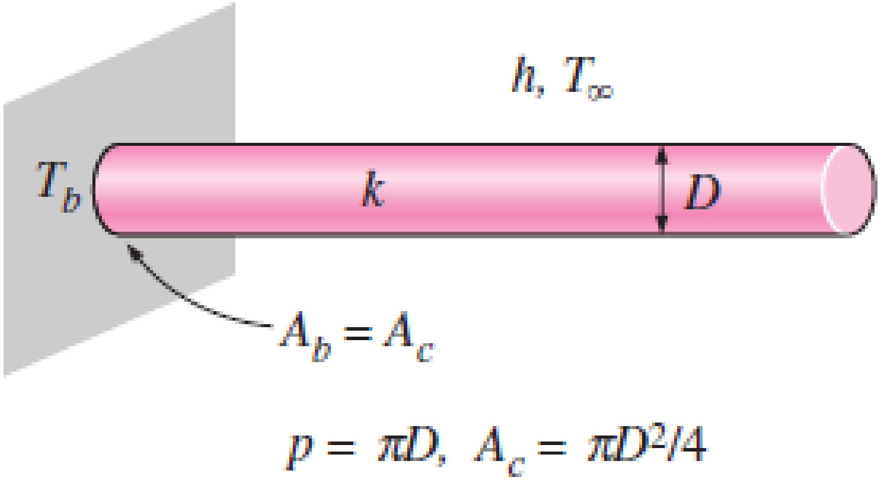
The design of a fin thus becomes an open-ended matter of optimizing, subject to many factors. Some of the factors that have to be considered include:

Fin Design

- The weight of material added by the fin. This might be a cost factor or it might be an important consideration in this own right.
- The possible dependence of h on $(T - T_\infty)$, flow velocity past the fin, or other influences
- The influence of the fin (or fins) on the heat transfer coefficient, h, as the fluid moves around it (or them)
- The geometric configuration of the channel that the fin lies in
- The cost and complexity of manufacturing fins
- The pressure drop introduced by the fins

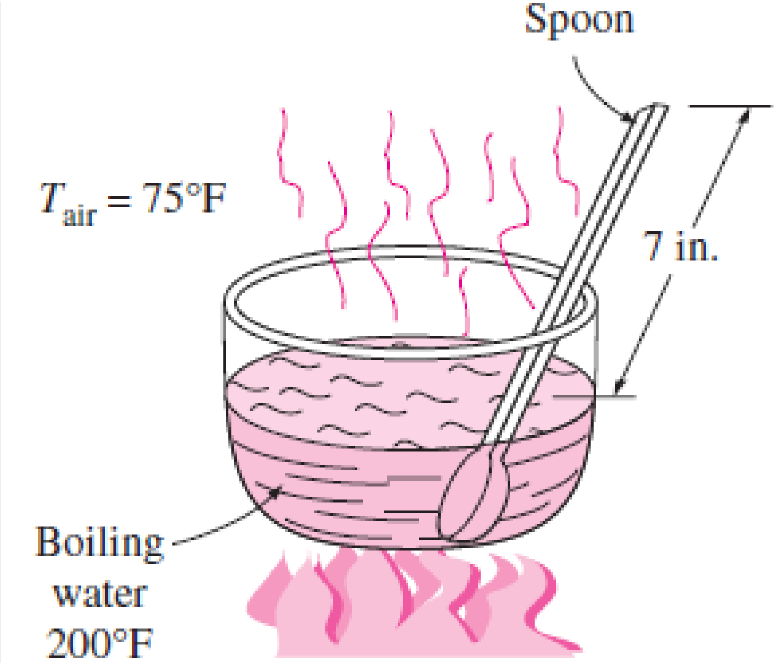
Exercise, Cengel

3–106 Obtain a relation for the fin efficiency for a fin of constant cross-sectional area A_c, perimeter p, length L, and thermal conductivity k exposed to convection to a medium at T_∞ with a heat transfer coefficient h. Assume the fins are sufficiently long so that the temperature of the fin at the tip is nearly T_∞. Take the temperature of the fin at the base to be T_b and neglect heat transfer from the fin tips. Simplify the relation for (*a*) a circular fin of diameter D and (*b*) rectangular fins of thickness t.



Example, Cengel

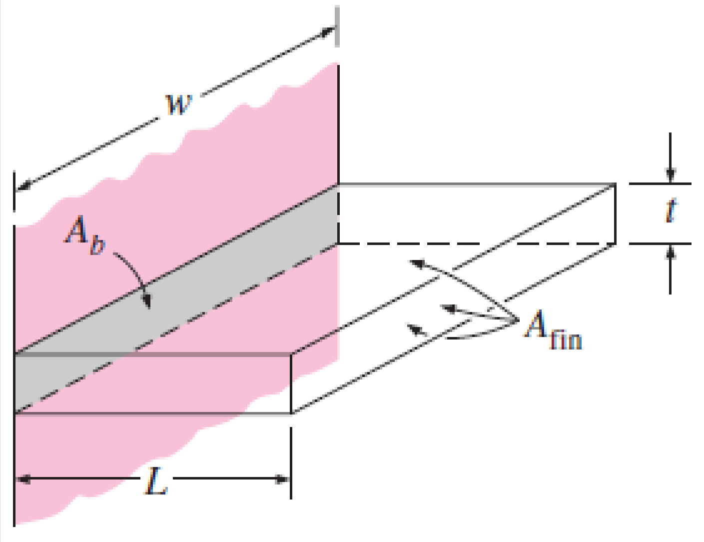
3–111E Consider a stainless steel spoon (k = 8.7 Btu/h · ft · °F) partially immersed in boiling water at 200°F in a kitchen at 75°F. The handle of the spoon has a cross section of 0.08 in. × 0.5 in., and extends 7 in. in the air from the free surface of the water. If the heat transfer coefficient at the exposed surfaces of the spoon handle is 3 Btu/h · ft^2 · °F, determine the temperature difference across the exposed surface of the spoon handle. State your assumptions. *Answer:* 124.6°F



Class work:

The extent to which the tip condition affects the thermal performance of a fin depends on the fin geometry and thermal conductivity, as well as the convection coefficient. Consider an alloyed aluminun (k=180 W/ m.K) rectangular fin whose base temperature is T_b= 100 °C. The fin is exposed to a fluid of temperature T_∞= 25°C, and the uniform convection coefficient of h= 100 W/m^2.K, may be assumed for the fin surface (tip condition).

* For a fin of length L= 10 mm, w= 5 mm, thickness t = 1 mm, determine the efficiency and effectiveness.



HEAT TRANSFER IN COMMON CONFIGURATIONS

So far, we have considered heat transfer in *simple* geometries such as large plane walls, long cylinders, and spheres.

This is because heat transfer in such geometries can be approximated as *one-dimensional*.

But many problems encountered in practice are two- or three-dimensional and involve rather complicated geometries for which no simple solutions are available.

An important class of heat transfer problems for which simple solutions are obtained encompasses those involving two surfaces maintained at *constant* temperatures T_1 and T_2.

The steady rate of heat transfer between these two surfaces is expressed as

$$Q = Sk(T_1 - T_2)$$

***S*: conduction shape factor**

k: the thermal conductivity of the medium between the surfaces

The conduction shape factor depends on the *geometry* of the system only. Conduction shape factors are applicable only when heat transfer between the two surfaces is by *conduction.*

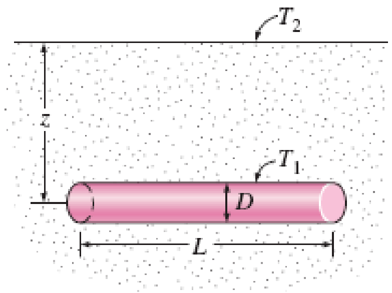
$$S = 1/kR$$

Relationship between the conduction shape factor and the thermal resistance

Conduction shape factors S for several configurations for use in $\dot{Q} = kS(T_1 - T_2)$ to determine the steady rate of heat transfer through a medium of thermal conductivity k between the surfaces at temperatures T_1 and T_2

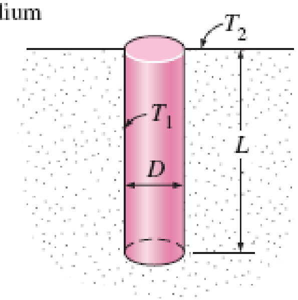
(1) Isothermal cylinder of length L buried in a semi-infinite medium ($L >> D$ and $z > 1.5D$)

$$S = \frac{2\pi L}{\ln(4z/D)}$$



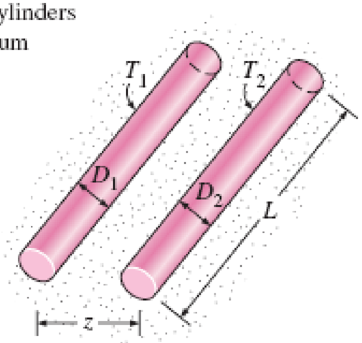
(2) Vertical isothermal cylinder of length L buried in a semi-infinite medium ($L >> D$)

$$S = \frac{2\pi L}{\ln(4L/D)}$$



(3) Two parallel isothermal cylinders placed in an infinite medium ($L >> D_1, D_2, z$)

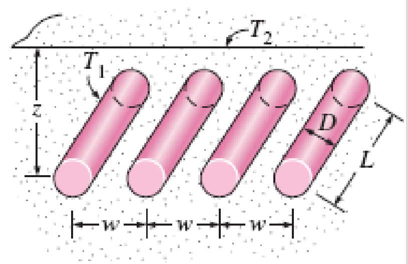
$$S = \frac{2\pi L}{\cosh^{-1}\left(\dfrac{4z^2 - D_1^2 - D_2^2}{2D_1D_2}\right)}$$



(4) A row of equally spaced parallel isothermal cylinders buried in a semi-infinite medium ($L >> D, z,$ and $w > 1.5D$)

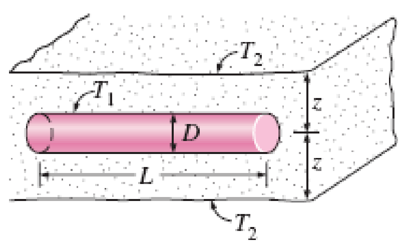
$$S = \frac{2\pi L}{\ln\left(\dfrac{2w}{\pi D}\sinh\dfrac{2\pi z}{w}\right)}$$

(per cylinder)



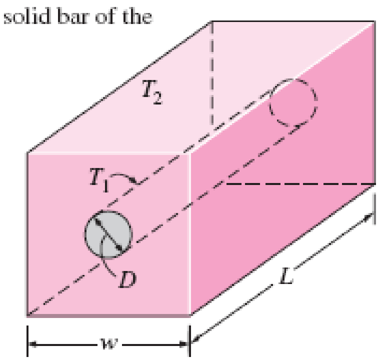
(5) Circular isothermal cylinder of length L in the midplane of an infinite wall ($z > 0.5D$)

$$S = \frac{2\pi L}{\ln(8z/\pi D)}$$



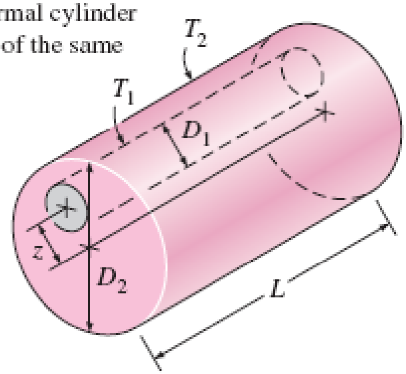
(6) Circular isothermal cylinder of length L at the center of a square solid bar of the same length

$$S = \frac{2\pi L}{\ln(1.08w/D)}$$



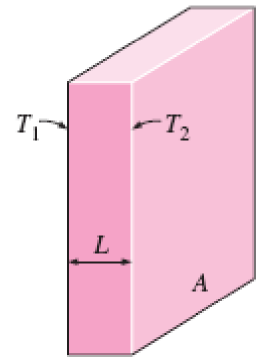
(7) Eccentric circular isothermal cylinder of length L in a cylinder of the same length ($L > D_2$)

$$S = \frac{2\pi L}{\cosh^{-1}\left(\dfrac{D_1^2 + D_2^2 - 4z^2}{2D_1D_2}\right)}$$



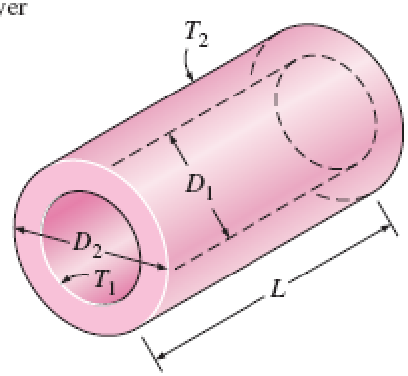
(8) Large plane wall

$$S = \frac{A}{L}$$



(9) A long cylindrical layer

$$S = \frac{2\pi L}{\ln(D_2/D_1)}$$



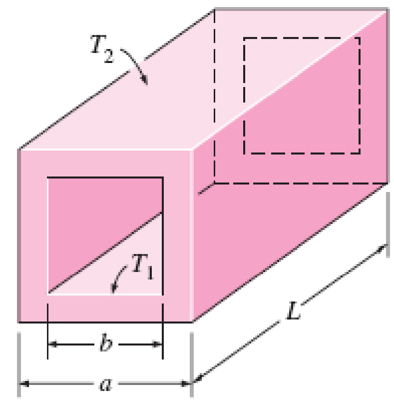
(10) A square flow passage

(a) For $a/b > 1.4$,

$$S = \frac{2\pi L}{0.93 \ln(0.948a/b)}$$

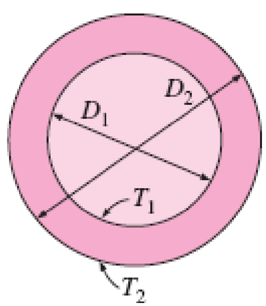
(b) For $a/b < 1.41$,

$$S = \frac{2\pi L}{0.785 \ln(a/b)}$$



(11) A spherical layer

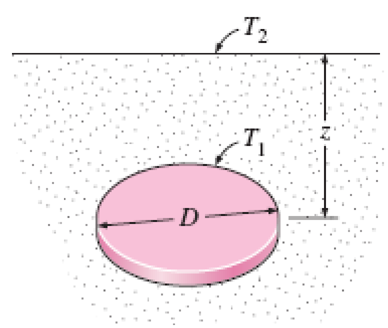
$$S = \frac{2\pi D_1 D_2}{D_2 - D_1}$$



(12) Disk buried parallel to the surface in a semi-infinite medium ($z >> D$)

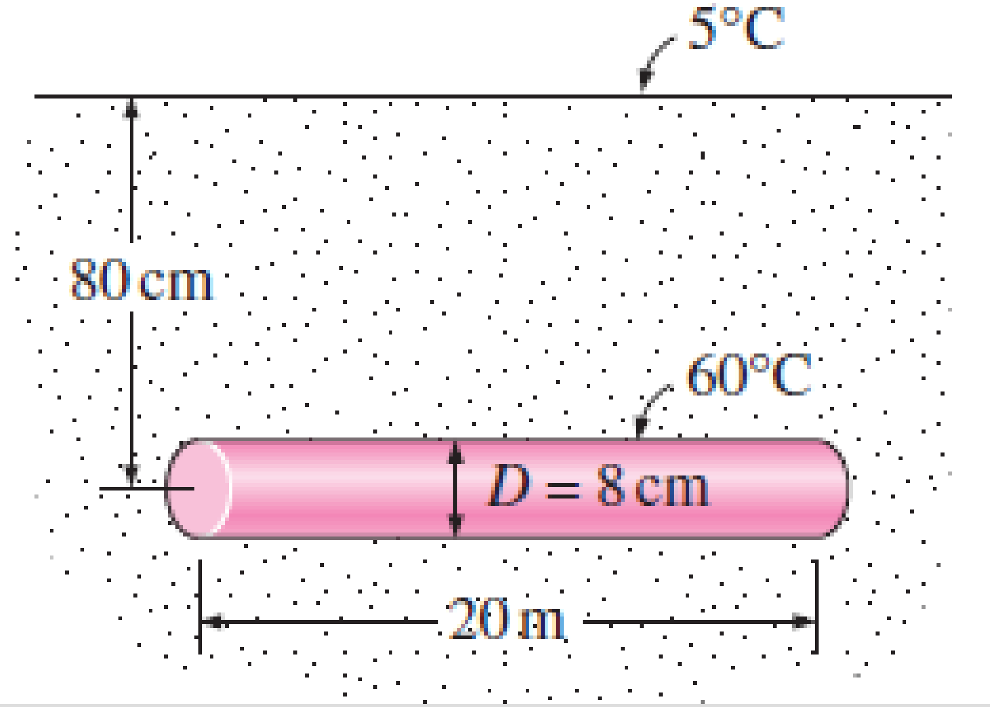
$S = 4D$

($S = 2D$ when $z = 0$)



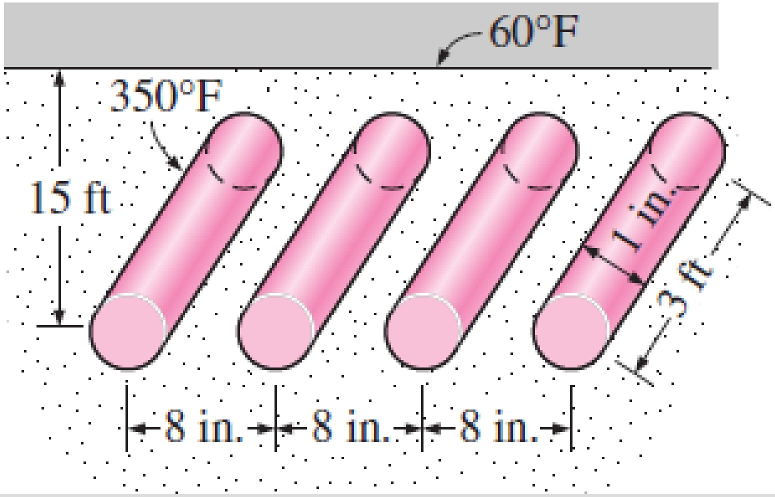
Example, Cengel

3-122 A 20-m-long and 8-cm-diameter hot water pipe of a district heating system is buried in the soil 80 cm below the ground surface. The outer surface temperature of the pipe is 60°C. Taking the surface temperature of the earth to be 5°C and the thermal conductivity of the soil at that location to be 0.9 W/m · °C, determine the rate of heat loss from the pipe.



Example, Cengel

3-126E A row of 3-ft-long and 1-in.-diameter used uranium fuel rods that are still radioactive are buried in the ground parallel to each other with a center-to-center distance of 8 in. at a depth 15 ft from the ground surface at a location where the thermal conductivity of the soil is 0.6 Btu/h · ft · °F. If the surface temperature of the rods and the ground are 350°F and 60°F, respectively, determine the rate of heat transfer from the fuel rods to the atmosphere through the soil.

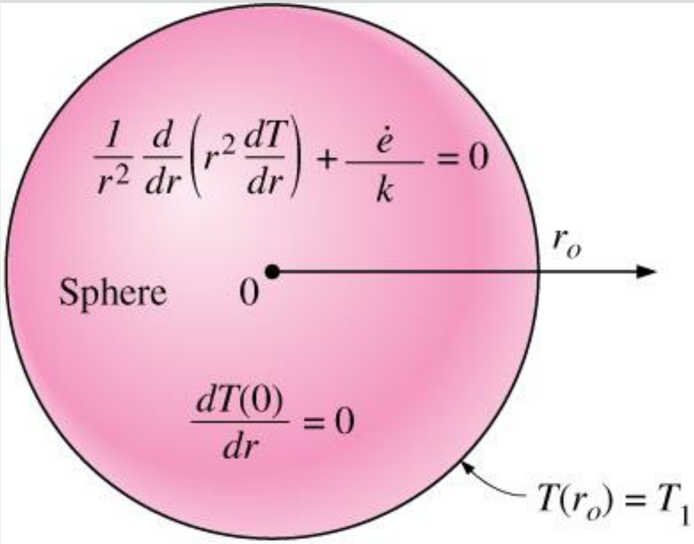


Fundamentals of Thermal-Fluid Sciences, 3rd Edition
Yunus A. Cengel, Robert H. Turner, John M. Cimbala
McGraw-Hill, 2008

Numerical Methods in Steady Heat conduction

Mehmet Kanoglu

Numerical Methods



Solution:

$$T(r) = T_1 + \frac{\dot{e}}{6k}(r_o^2 - r^2)$$

$$\dot{Q}(r) = -kA\frac{dT}{dr} = \frac{4\pi r^3 \dot{e}}{3}$$

FIGURE 5–1

The analytical solution of a problem requires solving the governing differential equation and applying the boundary conditions.

Limitations

Better modeling

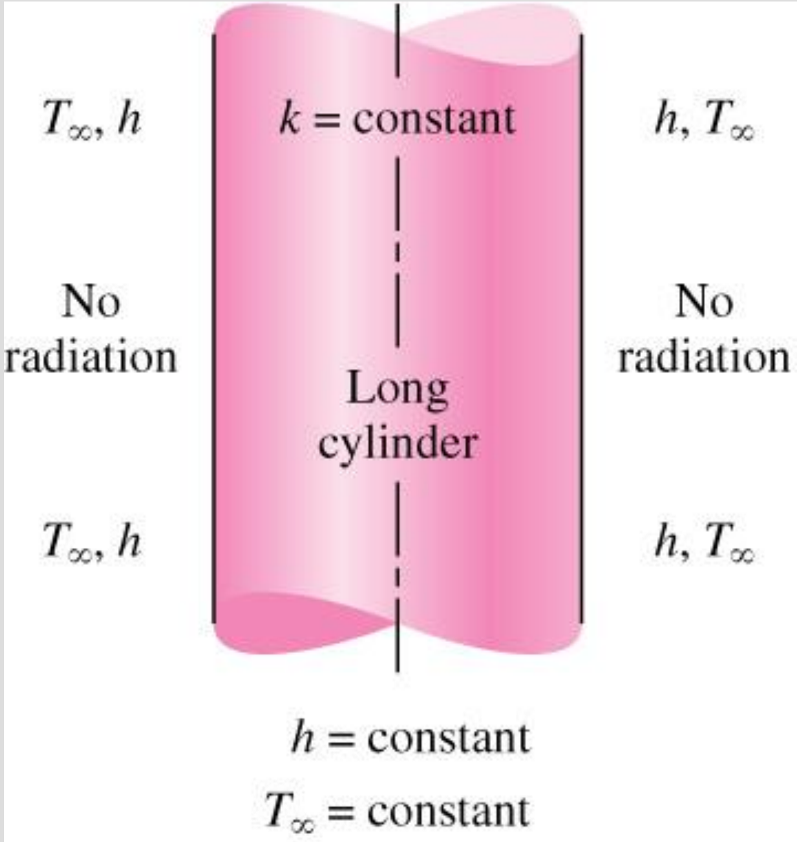


FIGURE 5–2

Analytical solution methods are limited to simplified problems in simple geometries.

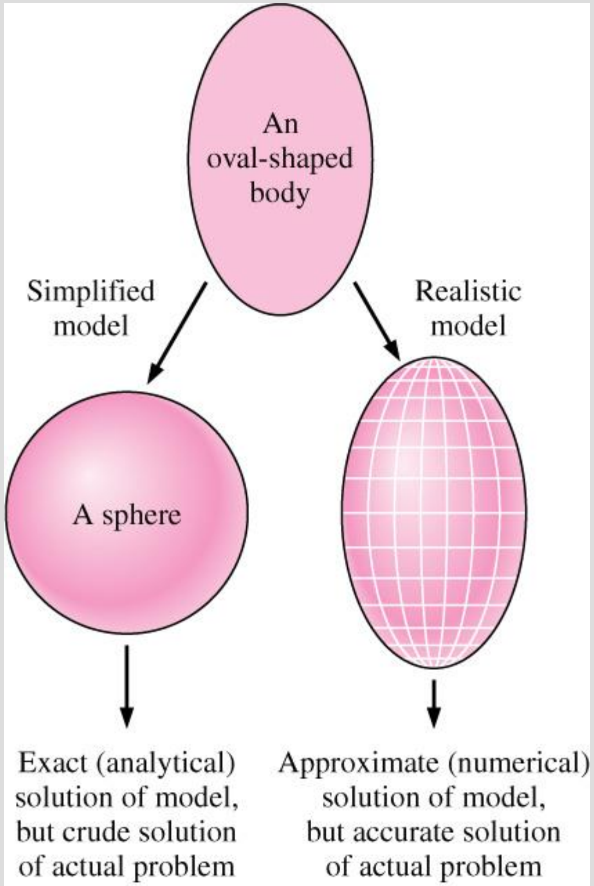


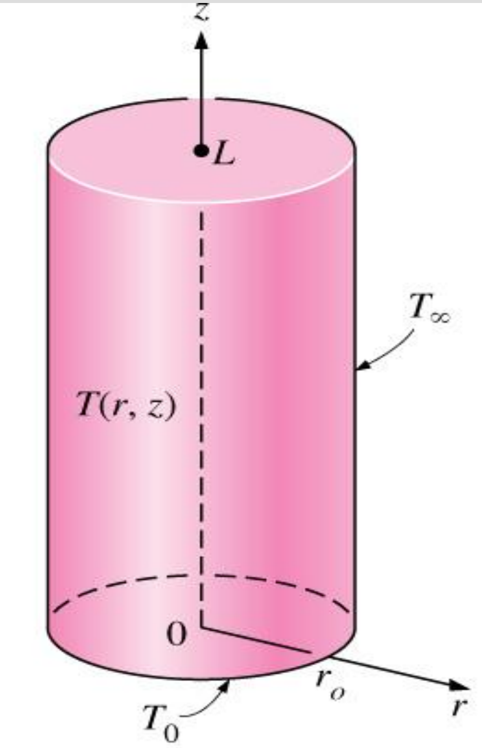
FIGURE 5–3

The approximate numerical solution of a real-world problem may be more accurate than the exact (analytical) solution of an oversimplified model of that problem.

Flexibility

Complications

Computers and numerical methods are ideally suited for such calculations, and a wide range of related problems can be solved by minor modifications in the code or input variables. Today it is almost unthinkable to perform any significant optimization studies in engineering without the power and flexibility of computers and numerical methods



Analytical solution:

$$\frac{T(r,z) - T_\infty}{T_0 - T_\infty} = \sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n r_o)} \frac{\sinh \lambda_n (L - z)}{\sinh (\lambda_n L)}$$

where λ_n's are roots of $J_0(\lambda_n r_o) = 0$

FIGURE 5–4

Some analytical solutions are very complex and difficult to use.

Human Nature

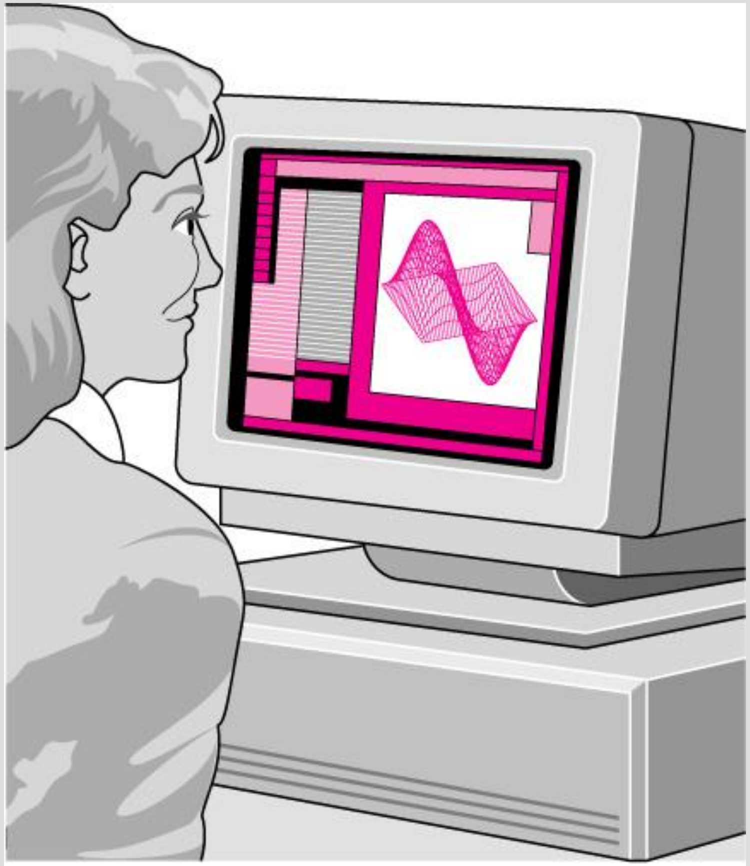
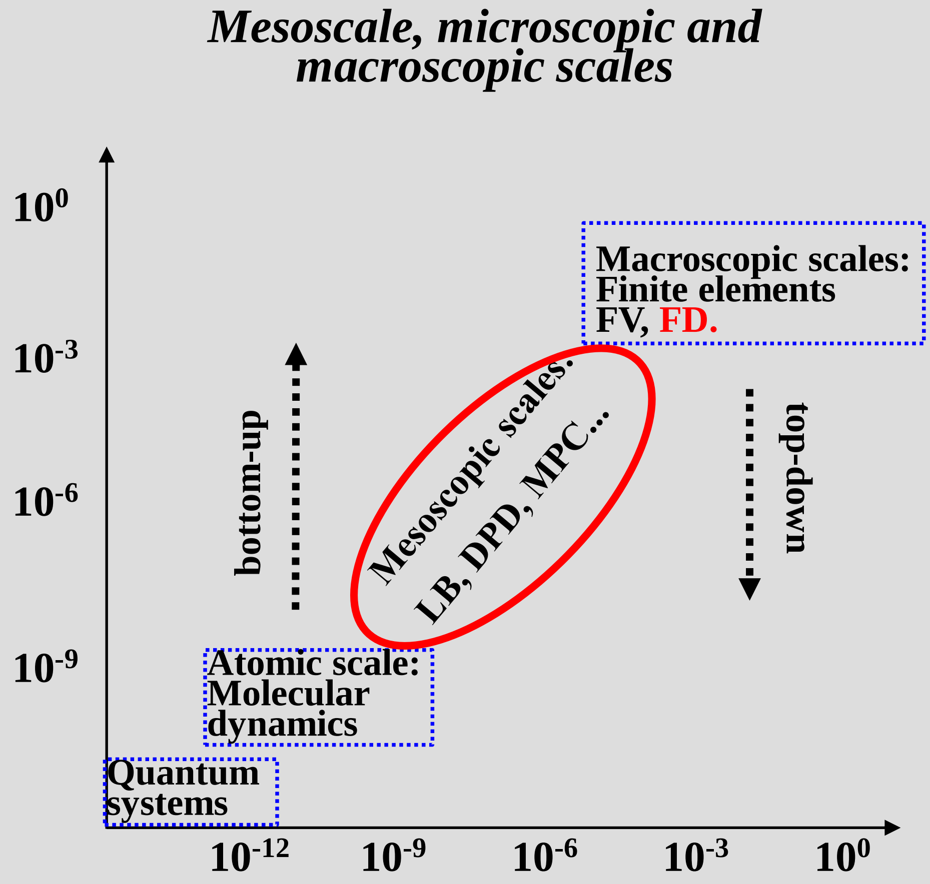


FIGURE 5–5
The ready availability of high-powered computers with sophisticated software packages has made numerical solution the norm rather than the exception.

Numerical Simulation

scale of simulation:



Numerical Simulation

In the past, two approaches in science:

- Theoretical
- Experimental

Computer $\Longrightarrow$ $\boxed{\text{Numerical simulation}}$

Expensive experiments are being replaced by numerical simulations :

- cheaper and faster
- simulation of phenomena that can not be experimentally reproduced (weather, ocean, ...)

Numerical Simulation

Navier-Stokes equations analytically solvable only in special cases

⇨ approximate the solution numerically

use a discretization method to approximate the differential equations by a system of algebraic equations which can be solved on a computer

- Finite Differences (FD)
- Finite Volume Method (FVM)
- Finite Element Method (FEM)

Numerical Simulation

Diferencias y analogías entre los métodos FDM-FEM-FVM:

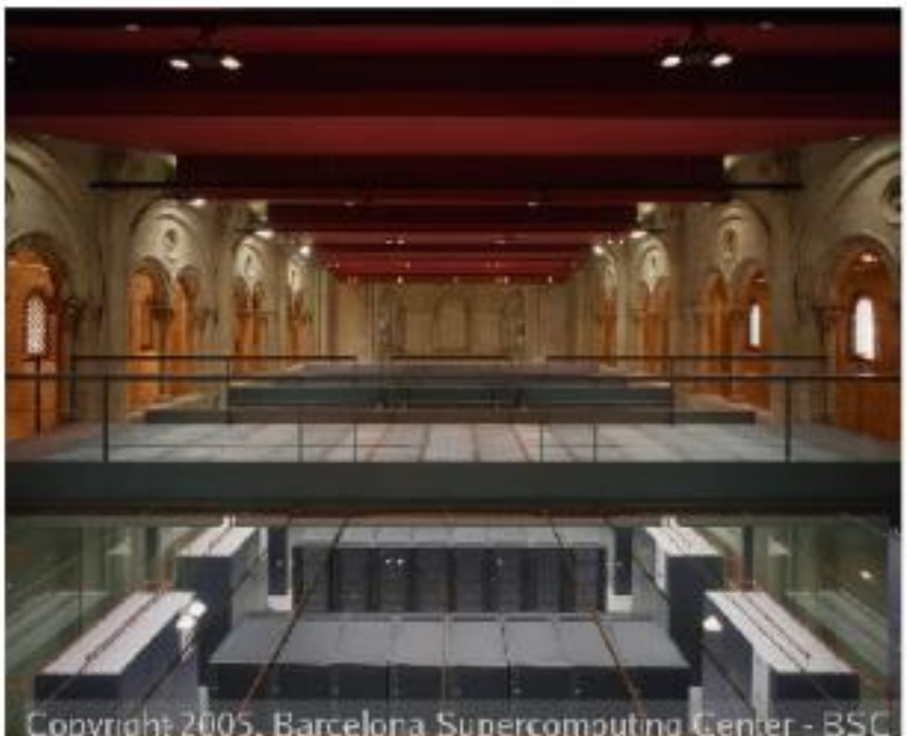
- **Métodos de diferencias finitas**: FDM
  1. Fácil de formular.
  2. En problemas multidimensionales, la malla debe ser estructurada en las tres direcciones espaciales. Las mallas curvas deben transformarse a coordenadas cartesianas de manera que las EDPs se reescriben en dicho sistema de referencia cartesiano.
  3. Las CCs tipo Neumann se imponen de manera aproximada, pero no exacta.
- **Métodos de elementos finitos**: FEM
  1. Los principios y la formulación del método exigen el rigor matemático.
  2. Las geometrías complejas y las mallas no estructuradas se pueden tratar de manera trivial.
  3. Las condiciones de contorno tipo Neumann se imponen de manera exacta.
- **Métodos de voúmenes finitos**: FVM
  1. Existe una formulación equivalente a FDM y FEM en mallas estructuradas.
  2. Las integrales de superficie y el uso de los flujos en las caras garantizan la propiedad de conservación.
  3. Las geometrías complejas y las mallas no estructuradas se tratan trivialmente sin necesidad de realizar cambios de coordenadas.

Numerical Simulation

- OPCIONES COMERCIALES
  - ANSYS FLUENT (http://www.ansys.com): FVM.
  - STAR-CCM+ (http://www.cd-adapco.com): FVM.
  - PHOENICS (http://www.cham.co.uk): FVM.
  - NaSt3DGPF (http://wissrech.ins.uni-bonn.de): FVM alto orden.
  - COSMOSFloWorks (http://www.cosmosm.com)
- InHouse: códigos propios de universidades y centros de investigación.
- OPCIONES LIBRES:
  - OPENFOAM (http://www.openfoam.com): FVM
  - FREECFD (http://www.freecfd.com): FVM
  - NaSt3DGP (http://wissrech.ins.uni-bonn.de/research/projects/NaSt3DGP/): FVM
  - FENICS (http://www.fenics.org): FEM
  - ELMER (http://www.csc.fi/english/pages/elmer): FEM
  - CLAWPACK (http://www.amath.washington.edu/ claw): FVM

Numerical Simulation

Barcelona
Supercomputing
Center
Centro Nacional de Supercomputación



Finite difference formulation of differential equations

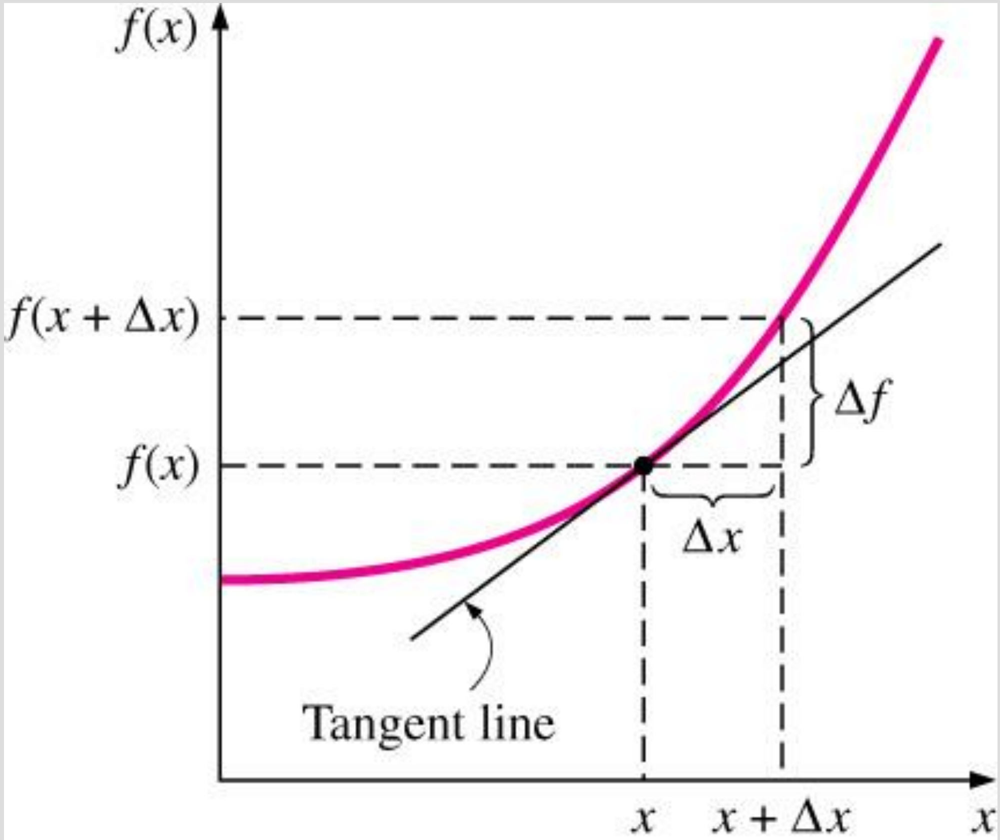


FIGURE 5–6

The derivative of a function at a point represents the slope of the function at that point.

Finite difference formulation of differential equations

The wall is subdivided into M sections of equal thickness $\Delta x = L/M$ in the x-direction, separated by planes passing through M+1 points *0, 1, 2, ..., m-1, m, m+1, ..., M* called nodes or nodal points. The x-coordinate of any point m is simply $x_m = m\Delta x$, and the temperature at the point is simply $T(x_m) = T_m$.

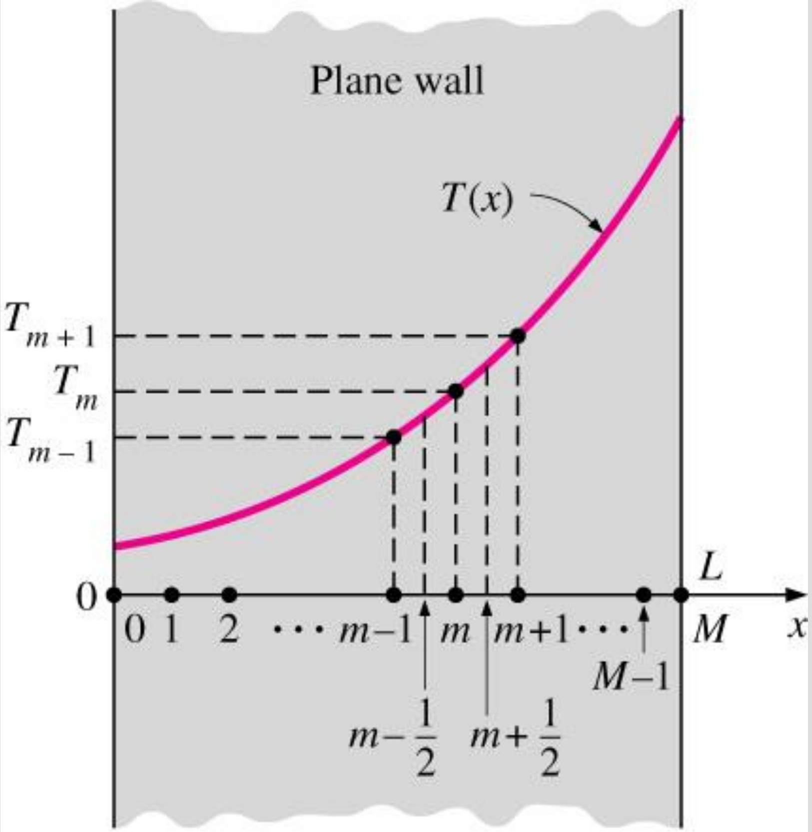


FIGURE 5–7

Schematic of the nodes and the nodal temperatures used in the development of the finite difference formulation of heat transfer in a plane wall.

Finite difference formulation of differential equations

The first derivate of temperature *dT/dx* at the midpoints *m-½* and *m+½* of the sections surrounding the node *m* can be expressed as

$$\left.\frac{dT}{dx}\right|_{m-\frac{1}{2}} \cong \frac{T_m - T_{m-1}}{\Delta x} \qquad \text{and} \qquad \left.\frac{dT}{dx}\right|_{m+\frac{1}{2}} \cong \frac{T_{m+1} - T_m}{\Delta x}$$

Noting that the second derivate is simply the derivate of the first derivate, the second derivate of temperature at node m can be expressed as

$$\left.\frac{d^2T}{dx^2}\right|_m \cong \frac{\left.\frac{dT}{dx}\right|_{m+\frac{1}{2}} - \left.\frac{dT}{dx}\right|_{m-\frac{1}{2}}}{\Delta x} = \frac{\frac{T_{m+1} - T_m}{\Delta x} - \frac{T_m - T_{m-1}}{\Delta x}}{\Delta x}$$

$$= \frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2}$$

Finite difference formulation of differential equations

The governing equation for *steady one-dimensional* heat transfer in a plane wall with heat generation and constant thermal conductivity, can be expressed in the *finite difference* form as

$$\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{g}_m}{k} = 0, \qquad m = 1, 2, 3, \ldots, M - 1$$

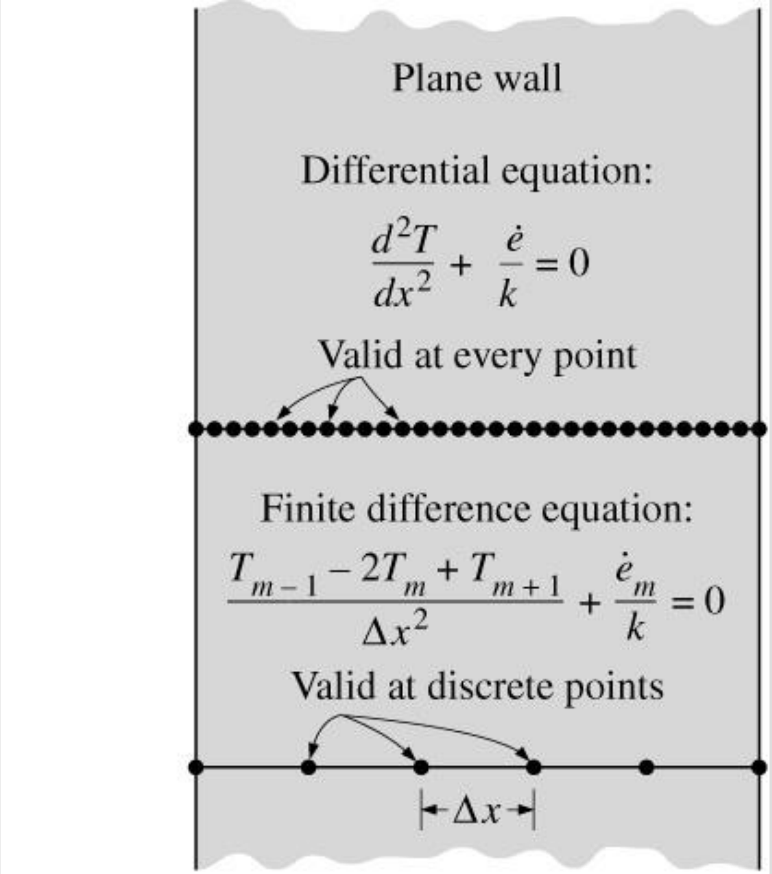


FIGURE 5–8
The differential equation is valid at every point of a medium, whereas the finite difference equation is valid at discrete points (the nodes) only.

Finite difference formulation of differential equations

The *finite difference formulation* for *steady two-dimensional heat conduction* in a region plane wall with heat generation and constant thermal conductivity, can be expressed in rectangular coordinates as

$$\frac{T_{m+1,n} - 2T_{m,n} + T_{m-1,n}}{\Delta x^2} +$$

$$+ \frac{T_{m,n+1} - 2T_{m,n} + T_{m,n-1}}{\Delta y^2} + \frac{\dot{g}_{m,n}}{k} = 0$$

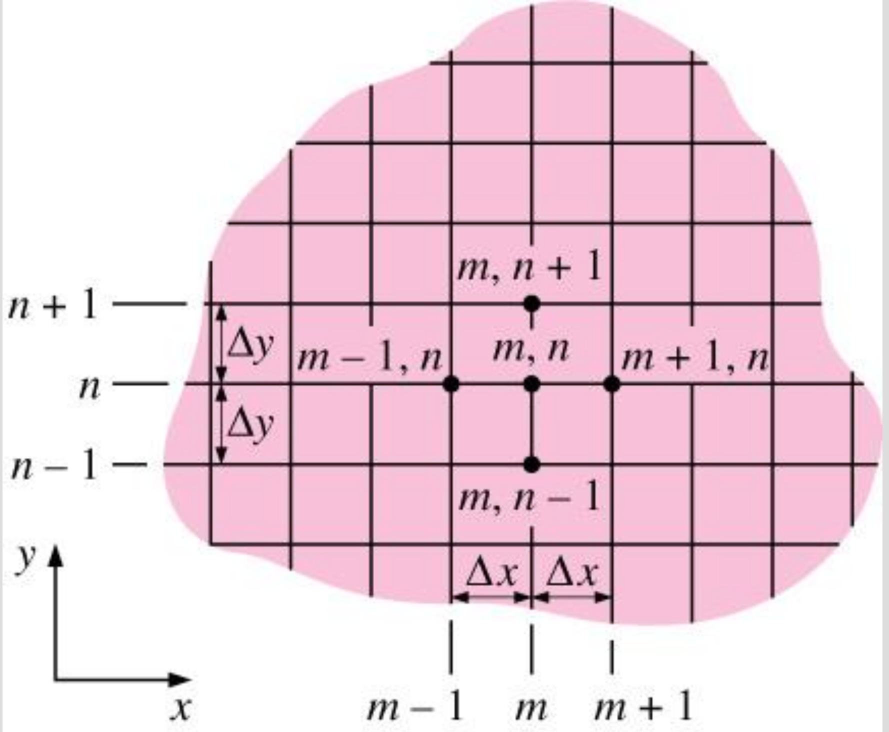


FIGURE 5–9
Finite difference mesh for two-dimensional conduction in rectangular coordinates.

One-dimensional steady heat conduction

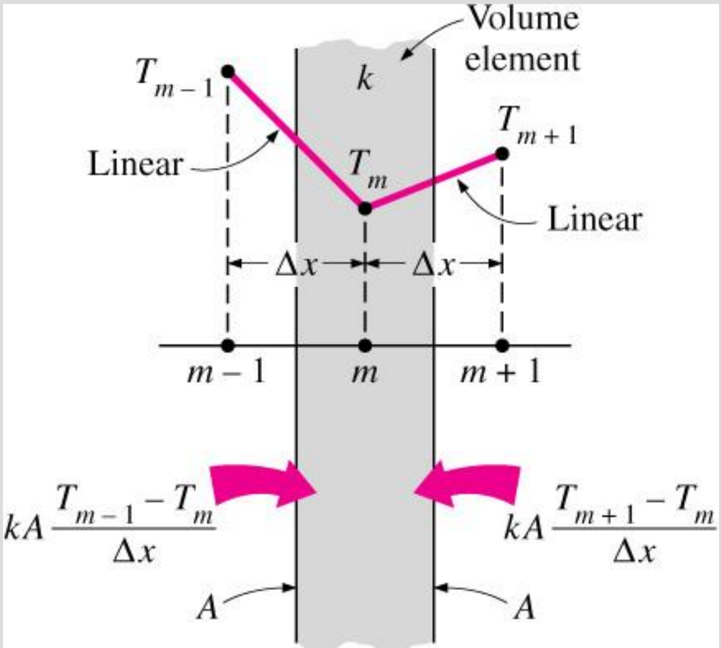


FIGURE 5–11

In finite difference formulation, the temperature is assumed to vary linearly between the nodes.

$$\begin{pmatrix}\text{Rate of heat}\\\text{conduction}\\\text{at the left}\\\text{surface}\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\\text{conduction}\\\text{at the right}\\\text{surface}\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\\text{generation}\\\text{inside the}\\\text{element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\\text{of the energy}\\\text{content of}\\\text{the element}\end{pmatrix}$$

$$\dot{Q}_{\text{cond, left}} + \dot{Q}_{\text{cond, right}} + \dot{G}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t} = 0$$

$$\dot{G}_{\text{element}} = \dot{g}_m V_{\text{element}} = \dot{g}_m A\Delta x$$

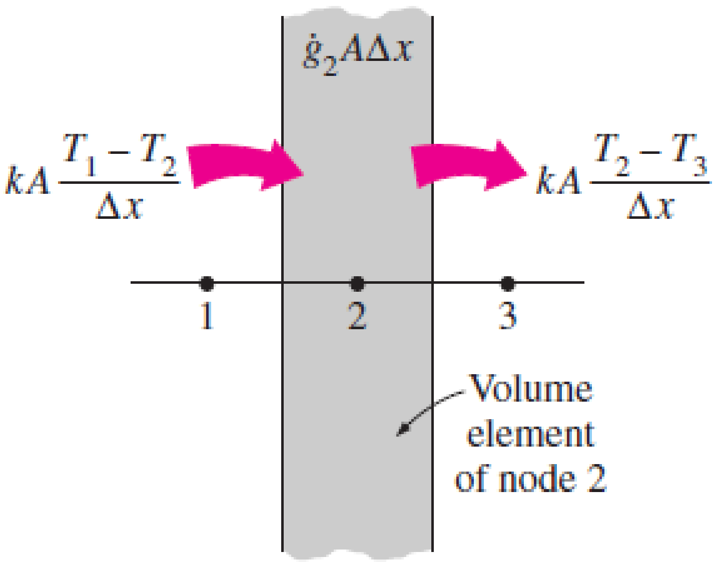
$$\dot{Q}_{\text{cond, left}} = kA\frac{T_{m-1} - T_m}{\Delta x}$$

$$\dot{Q}_{\text{cond, right}} = kA\frac{T_{m+1} - T_m}{\Delta x}$$

$$\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{g}_m}{k} = 0,$$

$$m = 1, 2, 3, \ldots, M - 1$$

One-dimensional steady heat conduction

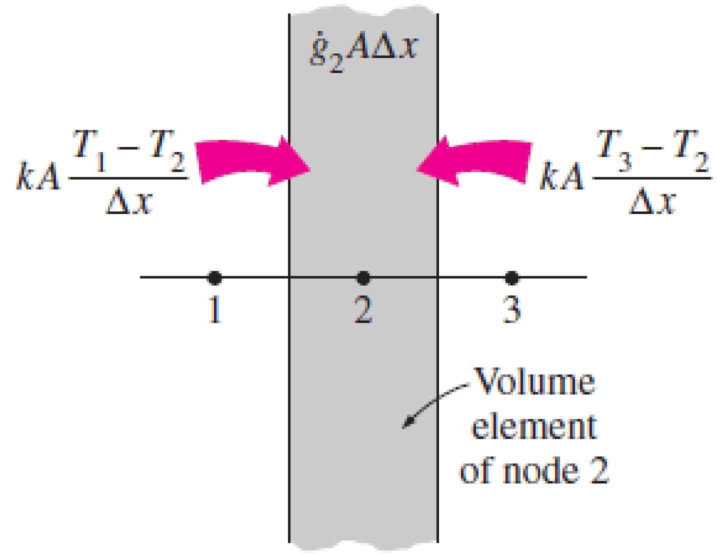


$$kA\frac{T_1 - T_2}{\Delta x} - kA\frac{T_2 - T_3}{\Delta x} + \dot{g}_2 A\Delta x = 0$$

or

$$T_1 - 2T_2 + T_3 + \dot{g}_2 A\Delta x^2 / k = 0$$

(*a*) Assuming heat transfer to be out of the volume element at the right surface.



$$kA\frac{T_1 - T_2}{\Delta x} + kA\frac{T_3 - T_2}{\Delta x} + \dot{g}_2 A\Delta x = 0$$

or

$$T_1 - 2T_2 + T_3 + \dot{g}_2 A\Delta x^2 / k = 0$$

(*b*) Assuming heat transfer to be into the volume element at all surfaces.

Boundary conditions

Boundary conditions most commonly encountered in practice are the *specified temperature, specified heat flux, convection,* and *radiation* boundary conditions

$$T(0) = T_0 = \text{Specified value}$$
$$T(L) = T_M = \text{Specified value}$$

$$\sum_{\text{all sides}} \dot{Q} + \dot{G}_{\text{element}} = 0$$

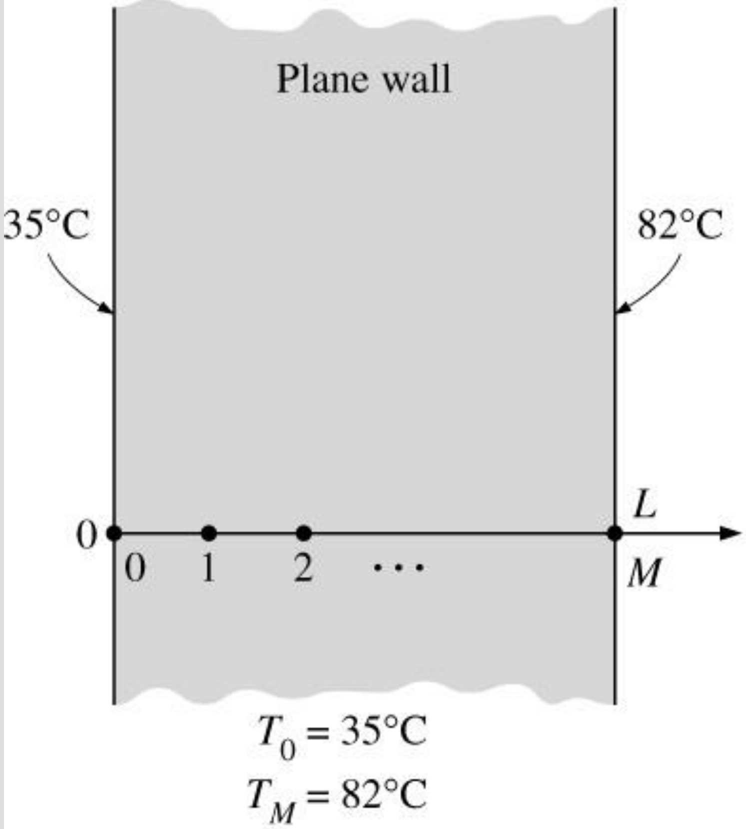


FIGURE 5–13
Finite difference formulation of specified temperature boundary conditions on both surfaces of a plane wall.

Boundary conditions

1. Specified Heat Flux Boundary Condition

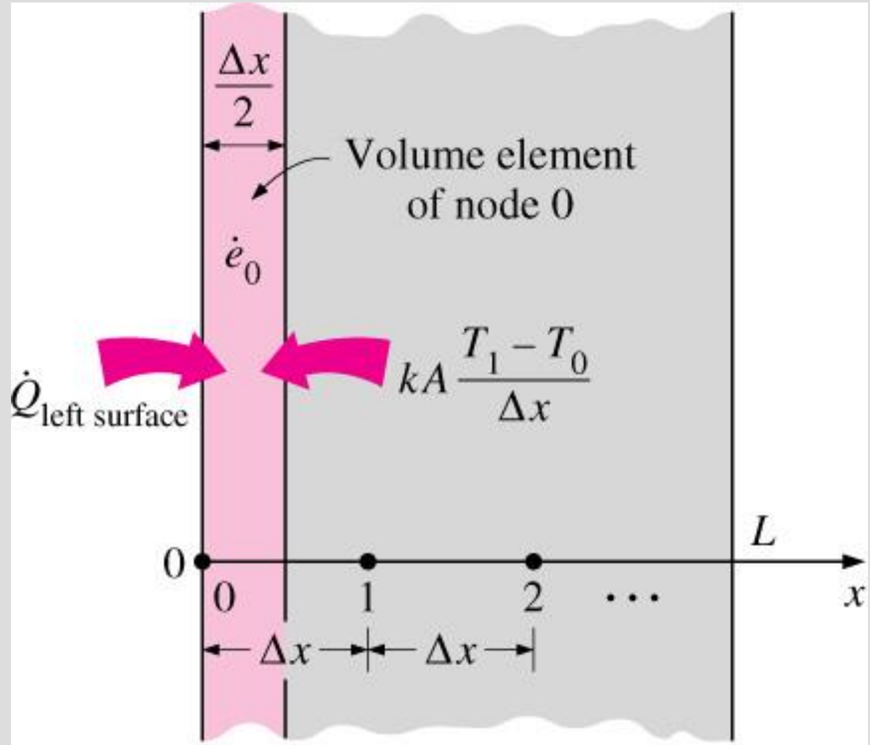
$$\dot{q}_0 A + kA\frac{T_1 - T_0}{\Delta x} + \dot{g}_0(A\Delta x/2) = 0$$

Special case: **Insulated Boundary**

$$kA\frac{T_1 - T_0}{\Delta x} + \dot{g}_0(A\Delta x/2) = 0$$

2. Convection Boundary Condition

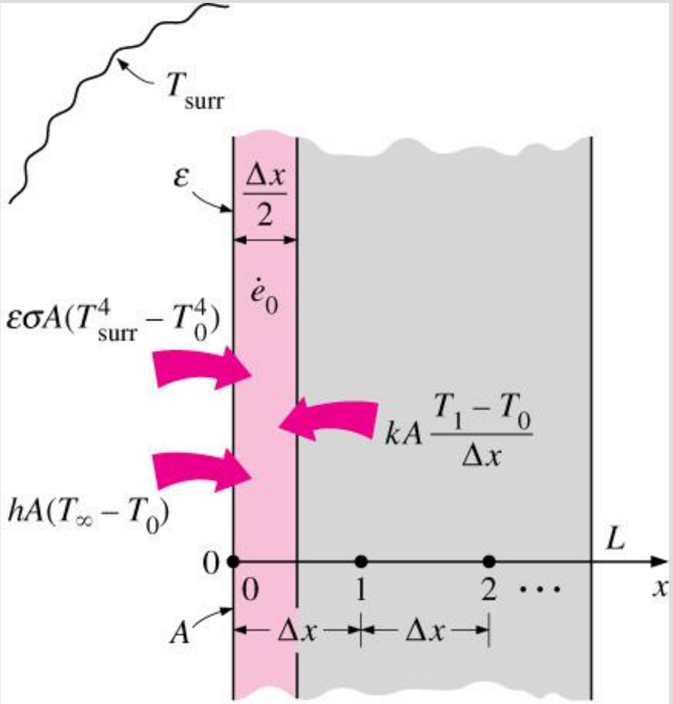
$$hA(T_\infty - T_0) + kA\frac{T_1 - T_0}{\Delta x} + \dot{g}_0(A\Delta x/2) = 0$$



$$\dot{Q}_{\text{left surface}} + kA\frac{T_1 - T_0}{\Delta x} + \dot{e}_0 A\frac{\Delta x}{2} = 0$$

FIGURE 5–14
Schematic for the finite difference formulation of the left boundary node of a plane wall.

Boundary conditions



$$hA(T_\infty - T_0) + \varepsilon\sigma A(T_{surr}^4 - T_0^4) + kA\frac{T_1 - T_0}{\Delta x} + \dot{e}_0 A\frac{\Delta x}{2} = 0$$

FIGURE 5–15
Schematic for the finite difference formulation of combined convection and radiation on the left boundary of a plane wall.

3. Radiation Boundary Condition

$$\varepsilon\sigma A(T_{surr}^4 - T_0^4) + kA\frac{T_1 - T_0}{\Delta x} + \dot{g}_0(A\Delta x/2) = 0$$

4. Combined Convection and Radiation Boundary Condition

$$hA(T_\infty - T_0) + \varepsilon\sigma A(T_{surr}^4 - T_0^4) + kA\frac{T_1 - T_0}{\Delta x} + \dot{g}_0(A\Delta x/2) = 0$$

$$h_{combined}A(T_\infty - T_0) + kA\frac{T_1 - T_0}{\Delta x} + \dot{g}_0(A\Delta x/2) = 0$$

5. Combined Convection, Radiation and Heat Flux Boundary Condition

$$\dot{q}_0 A + hA(T_\infty - T_0) + \varepsilon\sigma A(T_{surr}^4 - T_0^4) + kA\frac{T_1 - T_0}{\Delta x} + \dot{g}_0(A\Delta x/2) = 0$$

Boundary conditions

6. Interface Boundary Condition

$$k_A A \frac{T_{m-1} - T_m}{\Delta x} + k_B A \frac{T_{m+1} - T_m}{\Delta x} +$$

$$+ \dot{g}_{A,m}(A\Delta x/2) + \dot{g}_{B,m}(A\Delta x/2) = 0$$

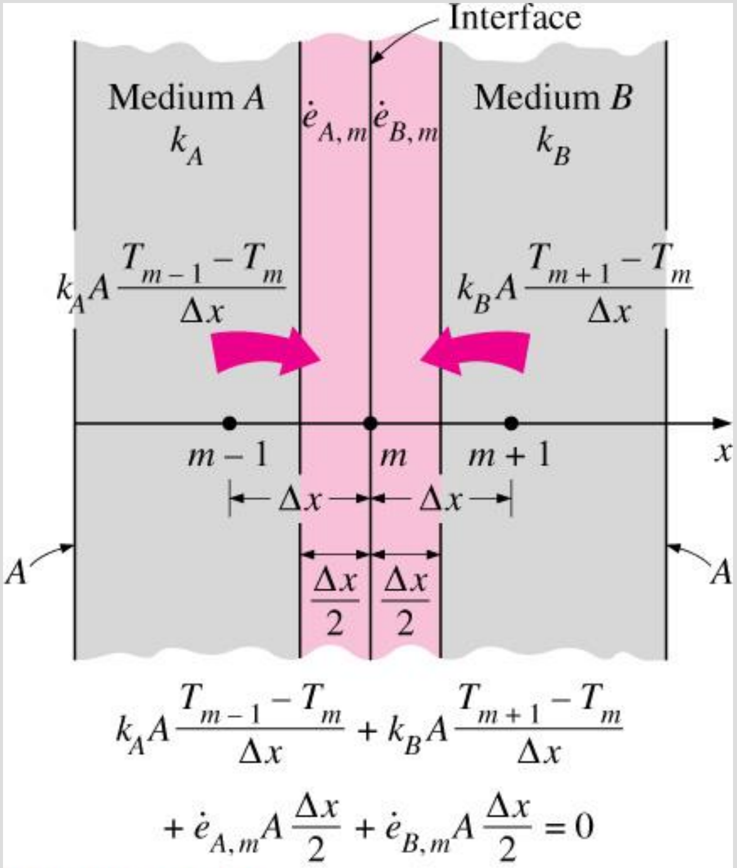


FIGURE 5–16
Schematic for the finite difference formulation of the interface boundary condition for two mediums *A* and *B* that are in perfect thermal contact.

Boundary conditions

Treating Insulated Boundary Nodes as Interior Nodes: The Mirror Image Concept

$$\frac{T_{m+1} - 2T_m + T_{m-1}}{\Delta x^2} + \frac{\dot{g}_m}{k} = 0$$

$$\frac{T_1 - 2T_0 + T_1}{\Delta x^2} + \frac{\dot{g}_0}{k} = 0$$

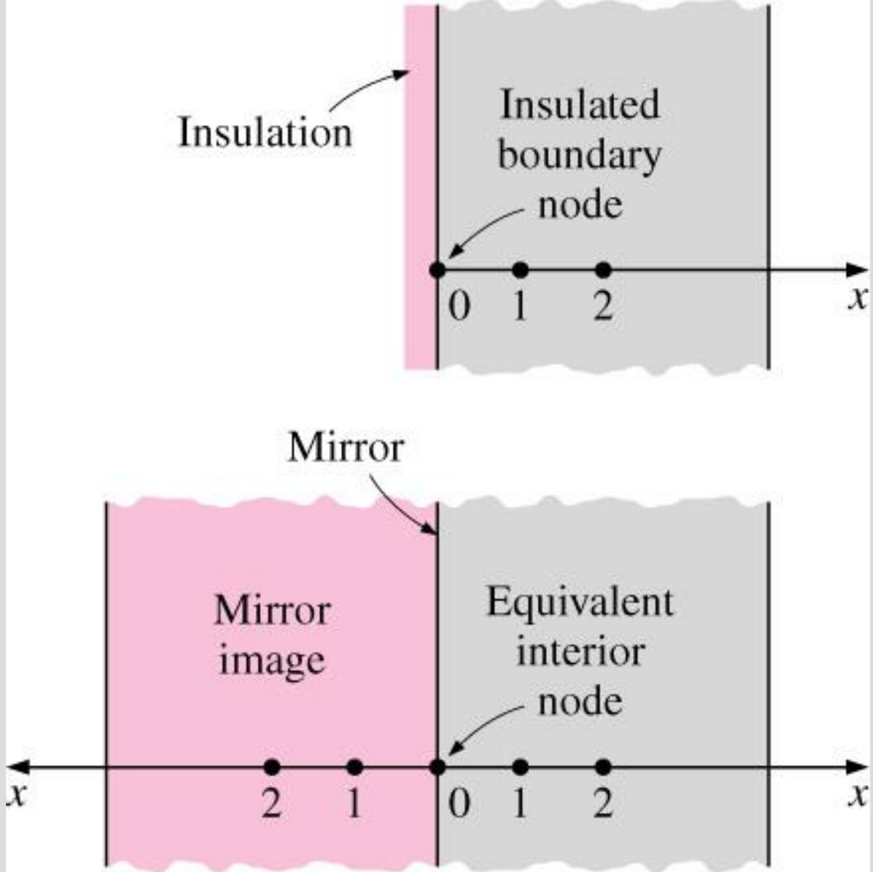
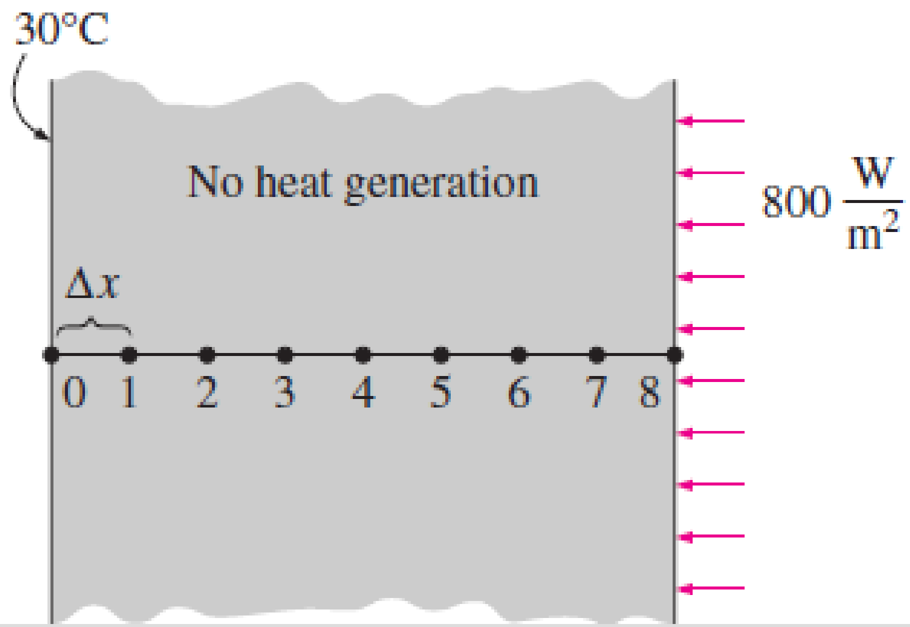


FIGURE 5–17

A node on an insulated boundary can be treated as an interior node by replacing the insulation by a mirror.

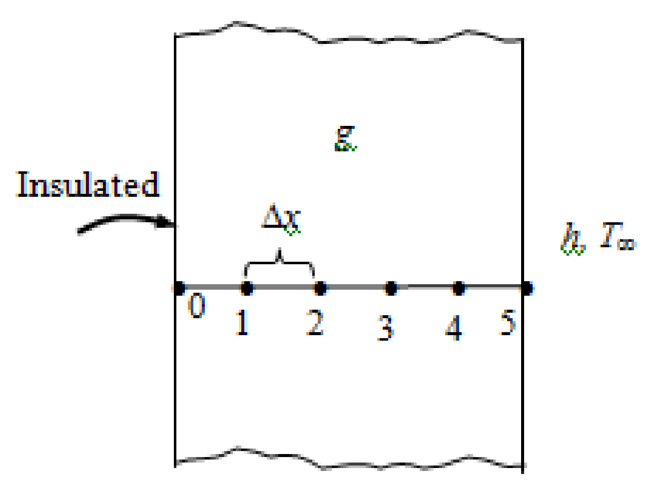
EXAMPLE

5–16 Consider steady heat conduction in a plane wall whose left surface (node 0) is maintained at 30°C while the right surface (node 8) is subjected to a heat flux of 800 W/m^2. Express the finite difference formulation of the boundary nodes 0 and 8



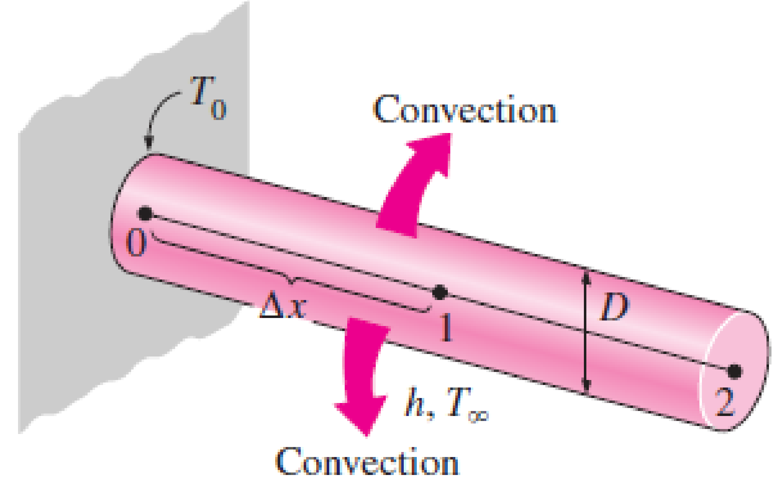
EXAMPLE

5–24 Consider a large uranium plate of thickness 5 cm and thermal conductivity $k = 28 \text{ W/m} \cdot °\text{C}$ in which heat is generated uniformly at a constant rate of $\dot{g} = 6 \times 10^5 \text{ W/m}^3$. One side of the plate is insulated while the other side is subjected to convection to an environment at 30°C with a heat transfer coefficient of $h = 60 \text{ W/m}^2 \cdot °\text{C}$. Considering six equally spaced nodes with a nodal spacing of 1 cm, (*a*) obtain the finite difference formulation of this problem and (*b*) determine the nodal temperatures under steady conditions by solving those equations.



QUIZ

Consider steady one-dimensional heat conduction in a pin fin of constant diameter D with constant thermal conductivity. The fin is losing heat by convection to the ambient air at T_∞ with a heat transfer coefficient of h. The nodal network of the fin consists of nodes 0 (at the base), 1 (in the middle), and 2 (at the fin tip) with a uniform nodal spacing of Δx. Using the energy balance approach, obtain the finite difference formulation of this problem to determine T_1 and T_2 for the case of specified temperature at the fin base and negligible heat transfer at the fin tip. All temperatures are in °C.



Two-Dimensional Steady Heat Conduction

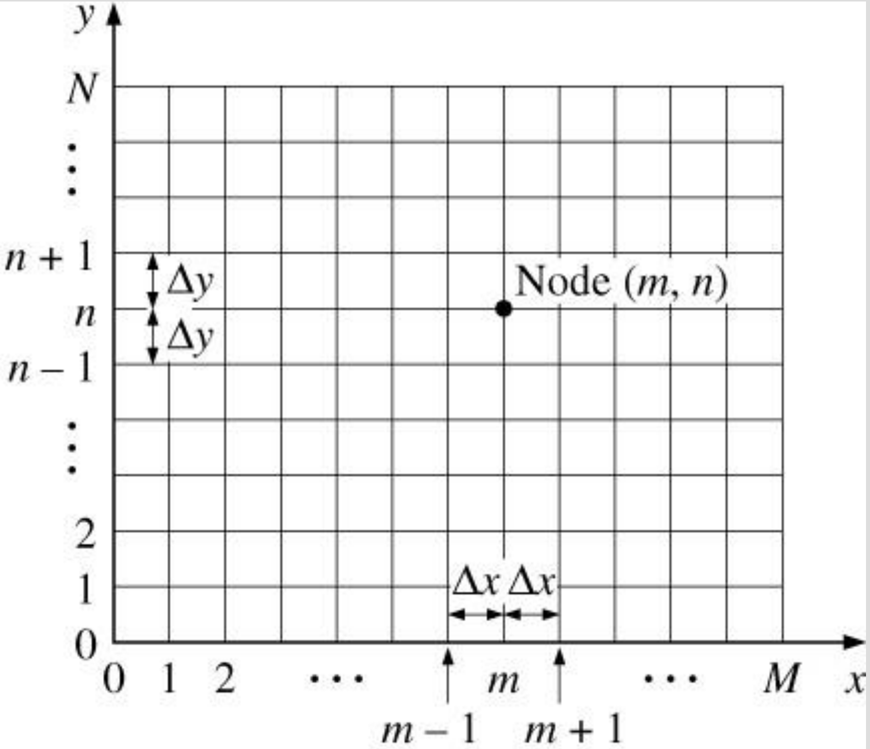


FIGURE 5–23

The nodal network for the finite difference formulation of two-dimensional conduction in rectangular coordinates.

A logical numbering scheme for two-dimensional problems is the *double subscript notation (m,n)* where *m = 0, 1, 2, ..., M* is the node count in the *x*-direction and *n = 0, 1, 2, ..., N* is the node count in the *y*-direction. The coordinates of the node *(m,n)* are simply $x = m\Delta x$ and $y = n\Delta y$, and the temperature at the node *(m,n)* is denoted by $T_{m,n}$.

Two-Dimensional Steady Heat Conduction

$$\left(\begin{array}{c}\text{Rate of heat conduction}\\ \text{at the left, top, right,}\\ \text{and bottom surfaces}\end{array}\right) + \left(\begin{array}{c}\text{Rate of heat}\\ \text{generation inside}\\ \text{the element}\end{array}\right) = \left(\begin{array}{c}\text{Rate of change of}\\ \text{the energy content}\\ \text{of the element}\end{array}\right)$$

$$\dot{Q}_{\text{cond, left}} + \dot{Q}_{\text{cond, top}} + \dot{Q}_{\text{cond, right}} + \dot{Q}_{\text{cond, bottom}} + \dot{G}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t} = 0$$

$$k\Delta y \frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k\Delta x \frac{T_{m,n+1} - T_{m,n}}{\Delta y} + k\Delta y \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

$$+ k\Delta x \frac{T_{m,n-1} - T_{m,n}}{\Delta y} + \dot{g}_{m,n}\,\Delta x\,\Delta y = 0$$

$$\frac{T_{m-1,n} - 2T_{m,n} + T_{m+1,n}}{\Delta x^2} + \frac{T_{m,n-1} - 2T_{m,n} + T_{m,n+1}}{\Delta y^2} + \frac{\dot{g}_{m,n}}{k} = 0$$

$$T_{m-1,n} + T_{m+1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n} + \frac{\dot{g}_{m,n} l^2}{k} = 0$$

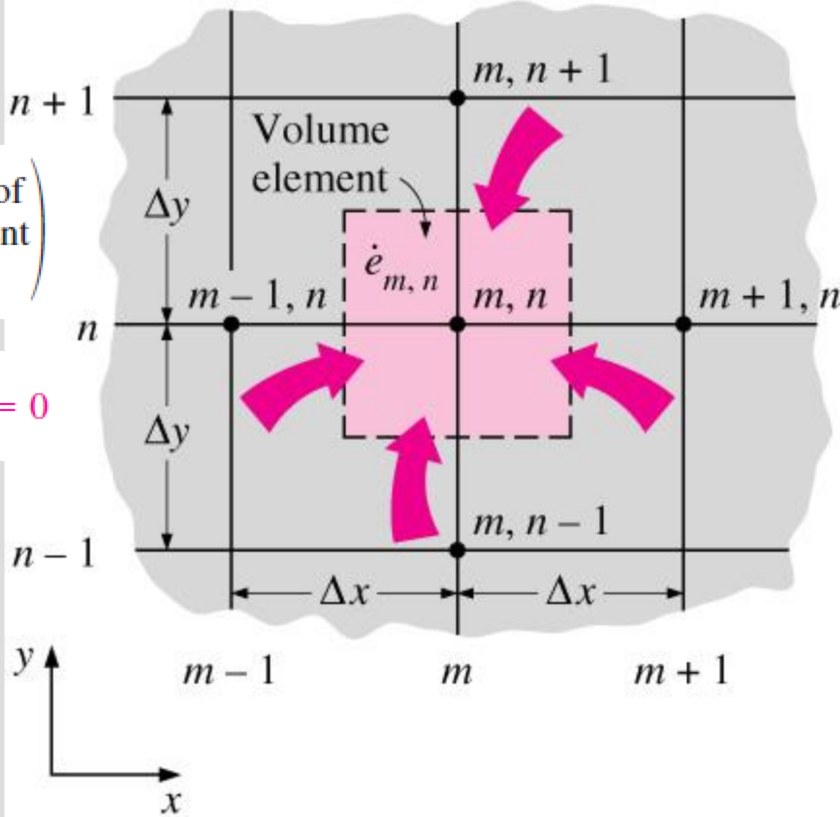
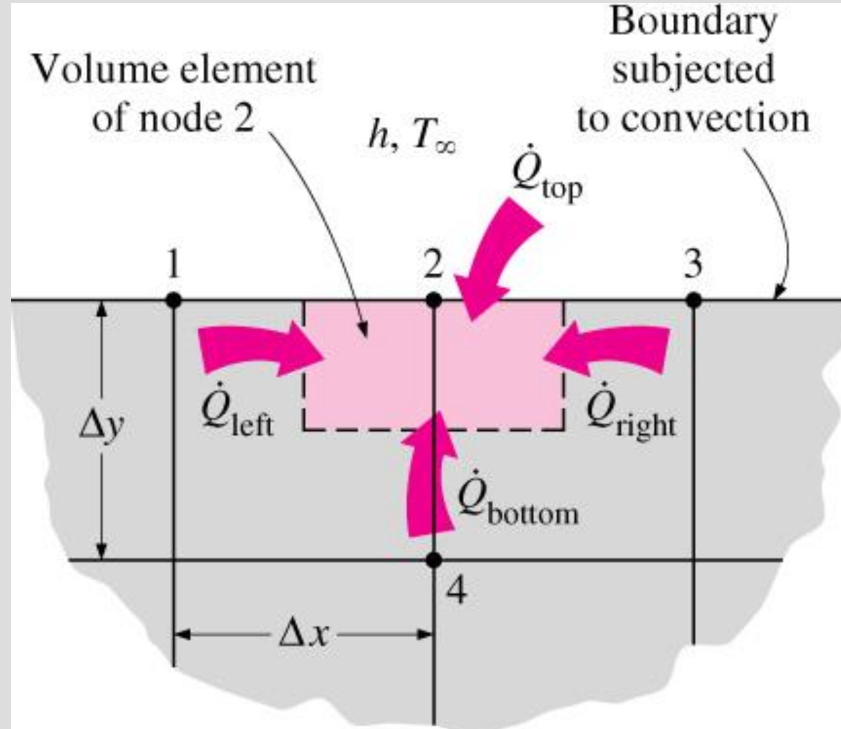


FIGURE 5–24
The volume element of a general interior node (m, n) for two-dimensional conduction in rectangular coordinates.

Boundary nodes

For the heat transfer under ***steady*** conditions, the basic equation to keep in mind when writing na ***energy balance*** on a volume element is

$$\sum_{\text{all sides}} \dot{Q} + \dot{g}V_{\text{element}} = 0$$

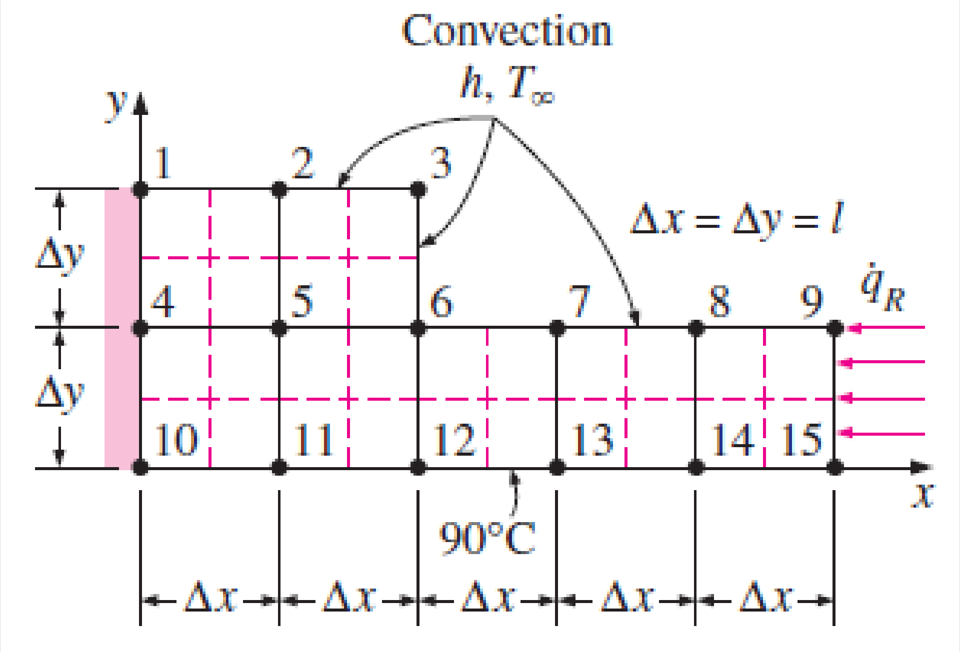


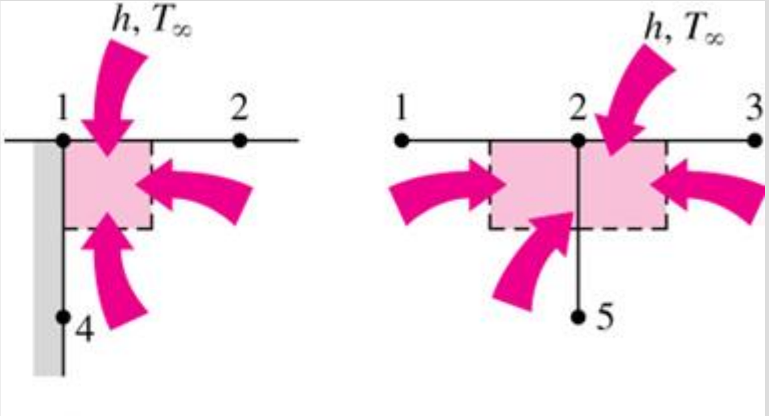
$$\dot{Q}_{\text{left}} + \dot{Q}_{\text{top}} + \dot{Q}_{\text{right}} + \dot{Q}_{\text{bottom}} + \frac{\dot{e}_2 V_2}{k} = 0$$

FIGURE 5–25
The finite difference formulation of a boundary node is obtained by writing an energy balance on its volume element.

Example:

Consider steady heat transfer in an L-shaped solid body whose cross section is given in Figure 5–26. Heat transfer in the direction normal to the plane of the paper is negligible, and thus heat transfer in the body is two-dimensional. The thermal conductivity of the body is $k = 15$ W/m · °C, and heat is generated in the body at a rate of $\dot{g} = 2 \times 10^6$ W/m^3. The left surface of the body is insulated, and the bottom surface is maintained at a uniform temperature of 90°C. The entire top surface is subjected to convection to ambient air at $T_\infty = 25$°C with a convection coefficient of $h = 80$ W/m^2 · °C, and the right surface is subjected to heat flux at a uniform rate of $\dot{q}_R = 5000$ W/m^2. The nodal network of the problem consists of 15 equally spaced nodes with $\Delta x = \Delta y = 1.2$ cm, as shown in the figure. Five of the nodes are at the bottom surface, and thus their temperatures are known. Obtain the finite difference equations at the remaining nine nodes and determine the nodal temperatures by solving them.





(*a*) Node 1 (*b*) Node 2

(*a*) *Node 1.* The volume element of this corner node is insulated on the left and subjected to convection at the top and to conduction at the right and bottom surfaces. An energy balance on this element gives [Fig. 5–27*a*]

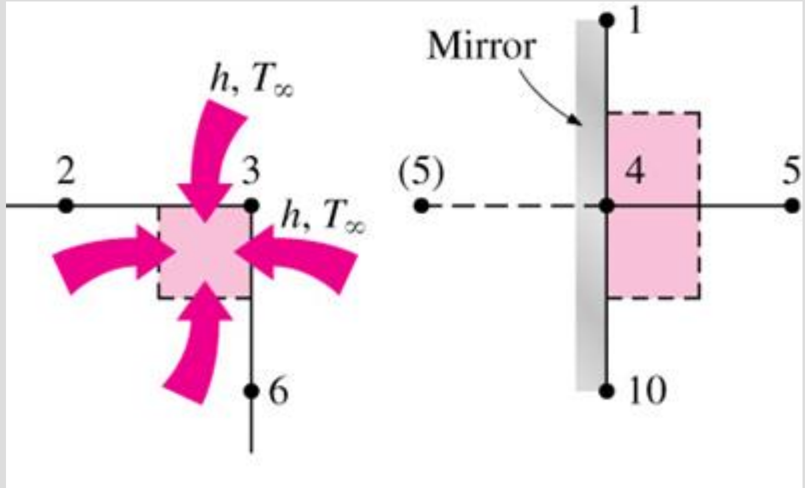
$$0 + h\frac{\Delta x}{2}(T_\infty - T_1) + k\frac{\Delta y}{2}\frac{T_2 - T_1}{\Delta x} + k\frac{\Delta x}{2}\frac{T_4 - T_1}{\Delta y} + \dot{g}_1\frac{\Delta x}{2}\frac{\Delta y}{2} = 0$$

$$-\left(2 + \frac{hl}{k}\right)T_1 + T_2 + T_4 = -\frac{hl}{k}T_\infty - \frac{\dot{g}_1 l^2}{2k}$$

(*b*) *Node 2.* The volume element of this boundary node is subjected to convection at the top and to conduction at the right, bottom, and left surfaces. An energy balance on this element gives [Fig. 5–27*b*]

$$h\Delta x(T_\infty - T_2) + k\frac{\Delta y}{2}\frac{T_3 - T_2}{\Delta x} + k\Delta x\frac{T_5 - T_2}{\Delta y} + k\frac{\Delta y}{2}\frac{T_1 - T_2}{\Delta x} + \dot{g}_2\Delta x\frac{\Delta y}{2} = 0$$

$$T_1 - \left(4 + \frac{2hl}{k}\right)T_2 + T_3 + 2T_5 = -\frac{2hl}{k}T_\infty - \frac{\dot{g}_2 l^2}{k}$$



(a) Node 3 (b) Node 4

(*c*) *Node 3.* The volume element of this corner node is subjected to convection at the top and right surfaces and to conduction at the bottom and left surfaces. An energy balance on this element gives [Fig. 5–28*a*]

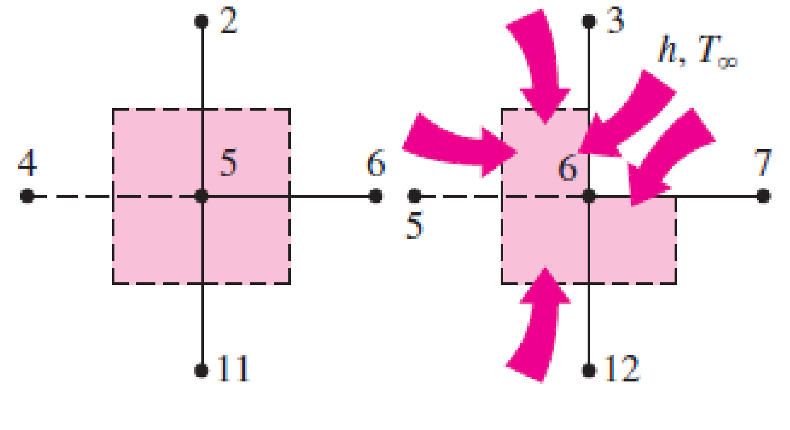
$$h\left(\frac{\Delta x}{2}+\frac{\Delta y}{2}\right)(T_\infty - T_3) + k\frac{\Delta x}{2}\frac{T_6 - T_3}{\Delta y} + k\frac{\Delta y}{2}\frac{T_2 - T_3}{\Delta x} + \dot{g}_3\frac{\Delta x}{2}\frac{\Delta y}{2} = 0$$

$$T_2 - \left(2 + \frac{2hl}{k}\right)T_3 + T_6 = -\frac{2hl}{k}T_\infty - \frac{\dot{g}_3 l^2}{2k}$$

(*d*) *Node 4.* This node is on the insulated boundary and can be treated as an interior node by replacing the insulation by a mirror. This puts a reflected image of node 5 to the left of node 4. Noting that $\Delta x = \Delta y = l$, the general interior node relation for the steady two-dimensional case (Eq. 5–35) gives [Fig. 5–28*b*]

$$T_5 + T_1 + T_5 + T_{10} - 4T_4 + \frac{\dot{g}_4 l^2}{k} = 0$$

$$T_1 - 4T_4 + 2T_5 = -90 - \frac{\dot{g}_4 l^2}{k}$$



(*a*) Node 5 (*b*) Node 6

(*e*) Node 5. This is an interior node, and noting that $\Delta x = \Delta y = l$, the finite difference formulation of this node is obtained directly from Eq. 5–35 to be [Fig. 5–29*a*]

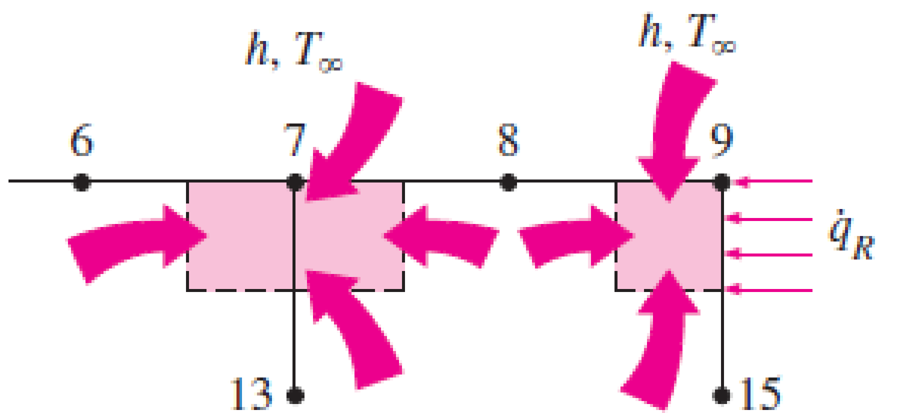
$$T_4 + T_2 + T_6 + T_{11} - 4T_5 + \frac{\dot{g}_5 l^2}{k} = 0$$

$$T_2 + T_4 - 4T_5 + T_6 = -90 - \frac{\dot{g}_5 l^2}{k}$$

(*f*) *Node 6.* The volume element of this inner corner node is subjected to convection at the L-shaped exposed surface and to conduction at other surfaces. An energy balance on this element gives [Fig. 5–29*b*]

$$h\left(\frac{\Delta x}{2} + \frac{\Delta y}{2}\right)(T_\infty - T_6) + k\frac{\Delta y}{2}\frac{T_7 - T_6}{\Delta x} + k\Delta x\frac{T_{12} - T_6}{\Delta y}$$
$$+ k\Delta y\frac{T_5 - T_6}{\Delta x} + k\frac{\Delta x}{2}\frac{T_3 - T_6}{\Delta y} + \dot{g}_6\frac{3\Delta x\Delta y}{4} = 0$$

$$T_3 + 2T_5 - \left(6 + \frac{2hl}{k}\right)T_6 + T_7 = -180 - \frac{2hl}{k}T_\infty - \frac{3\dot{g}_6 l^2}{2k}$$

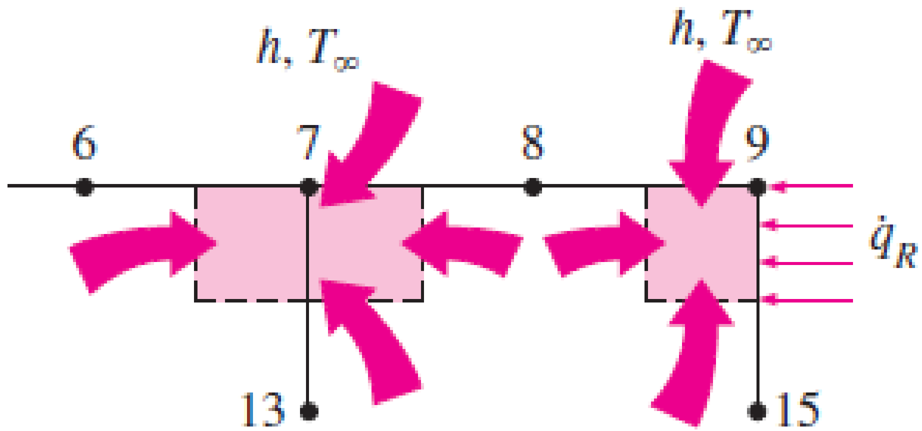


Schematics for energy balances on the volume elements of nodes 7 and 9.

(*g*) *Node 7.* The volume element of this boundary node is subjected to convection at the top and to conduction at the right, bottom, and left surfaces. An energy balance on this element gives [Fig. 5–30*a*]

$$h\Delta x(T_\infty - T_7) + k\frac{\Delta y}{2}\frac{T_8 - T_7}{\Delta x} + k\Delta x\frac{T_{13} - T_7}{\Delta y} + k\frac{\Delta y}{2}\frac{T_6 - T_7}{\Delta x} + \dot{g}_7\Delta x\frac{\Delta y}{2} = 0$$

$$T_6 - \left(4 + \frac{2hl}{k}\right)T_7 + T_8 = -180 - \frac{2hl}{k}T_\infty - \frac{\dot{g}_7 l^2}{k}$$



Schematics for energy balances on the volume elements of nodes 7 and 9.

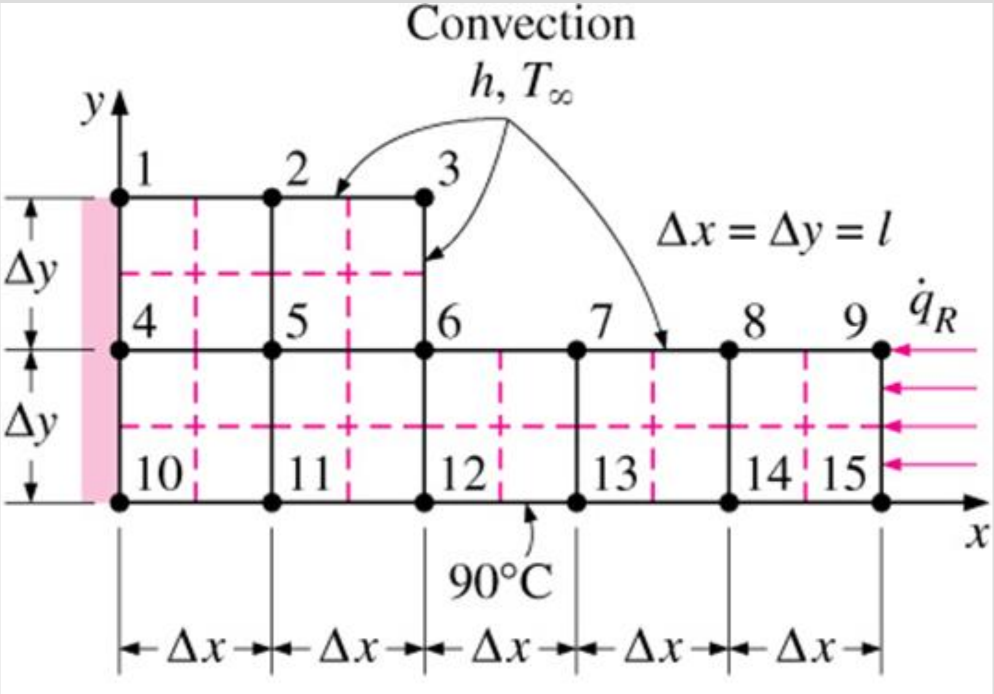
(*h*) *Node 8.* This node is identical to Node 7, and the finite difference formulation of this node can be obtained from that of Node 7 by shifting the node numbers by 1 (i.e., replacing subscript *m* by *m* + 1). It gives

$$T_7 - \left(4 + \frac{2hl}{k}\right) T_8 + T_9 = -180 - \frac{2hl}{k} T_\infty - \frac{\dot{g}_8 l^2}{k}$$

(*i*) *Node 9.* The volume element of this corner node is subjected to convection at the top surface, to heat flux at the right surface, and to conduction at the bottom and left surfaces. An energy balance on this element gives [Fig. 5–30*b*]

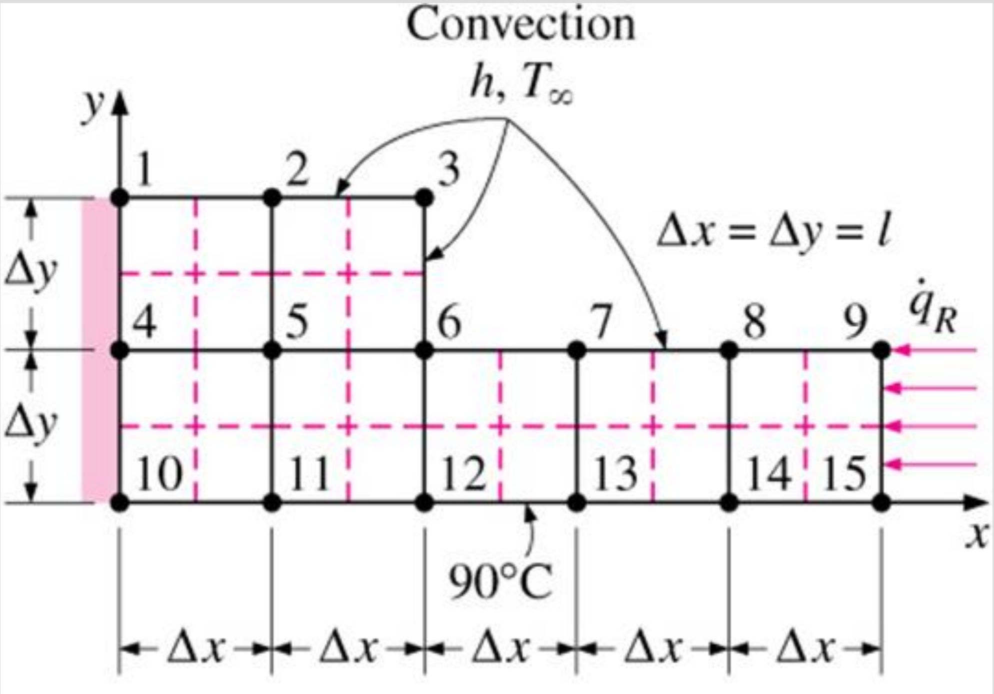
$$h\frac{\Delta x}{2}(T_\infty - T_9) + \dot{q}_R\frac{\Delta y}{2} + k\frac{\Delta x}{2}\frac{T_{15} - T_9}{\Delta y} + k\frac{\Delta y}{2}\frac{T_8 - T_9}{\Delta x} + \dot{g}_9\frac{\Delta x}{2}\frac{\Delta y}{2} = 0$$

$$T_8 - \left(2 + \frac{hl}{k}\right) T_9 = -90 - \frac{\dot{q}_R l}{k} - \frac{hl}{k} T_\infty - \frac{\dot{g}_9 l^2}{2k}$$



This completes the development of finite difference formulation for this problem. Substituting the given quantities, the system of nine equations for the determination of nine unknown nodal temperatures becomes

$$-2.064T_1 + T_2 + T_4 = -11.2$$
$$T_1 - 4.128T_2 + T_3 + 2T_5 = -22.4$$
$$T_2 - 2.128T_3 + T_6 = -12.8$$
$$T_1 - 4T_4 + 2T_5 = -109.2$$
$$T_2 + T_4 - 4T_5 + T_6 = -109.2$$
$$T_3 + 2T_5 - 6.128T_6 + T_7 = -212.0$$
$$T_6 - 4.128T_7 + T_8 = -202.4$$
$$T_7 - 4.128T_8 + T_9 = -202.4$$
$$T_8 - 2.064T_9 = -105.2$$



$T_1 = 112.1°C$	$T_2 = 110.8°C$	$T_3 = 106.6°C$
$T_4 = 109.4°C$	$T_5 = 108.1°C$	$T_6 = 103.2°C$
$T_7 = 97.3°C$	$T_8 = 96.3°C$	$T_9 = 97.6°C$

Irregular Boundaries

In problems with simple geometries, we can fill the entire region using simple volume elements such as strips for a plane wall and rectangular elements for two-dimensional conduction in a rectangular region. We can also use cylindrical or spherical shell elements to cover the cylindrical and spherical bodies entirely. However, many geometries encountered in practice such as turbine blades or engine blocks do not have simple shapes, and it is difficult to fill such geometries having irregular boundaries with simple volume elements. A practical way of dealing with such geometries is to replace the irregular geometry by a series of simple volume elements, as shown in Figure 5–31. This simple approach is often satisfactory for practical purposes, especially when the nodes are closely spaced near the boundary. More sophisticated approaches are available for handling irregular boundaries, and they are commonly incorporated into the commercial software packages.

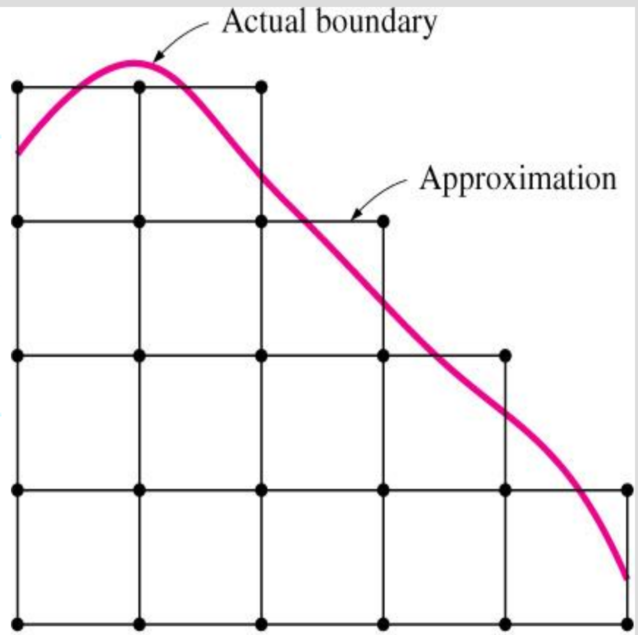
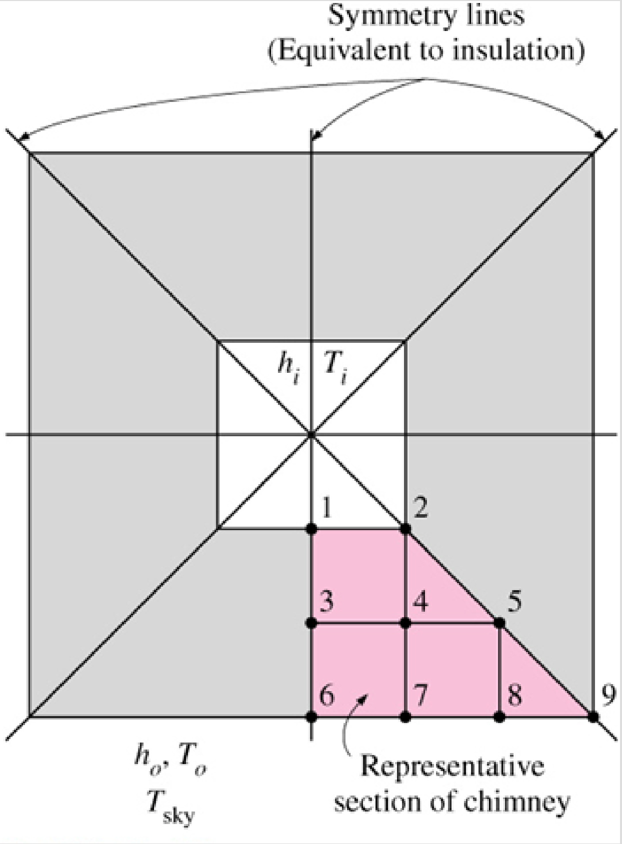


FIGURE 5–31
Approximating an irregular boundary with a rectangular mesh.

Symmetry sections

Analysis The cross section of the chimney is given in Figure 5–32. The most striking aspect of this problem is the apparent symmetry about the horizontal and vertical lines passing through the midpoint of the chimney as well as the diagonal axes, as indicated on the figure. Therefore, we need to consider only one-eighth of the geometry in the solution whose nodal network consists of nine equally spaced nodes.



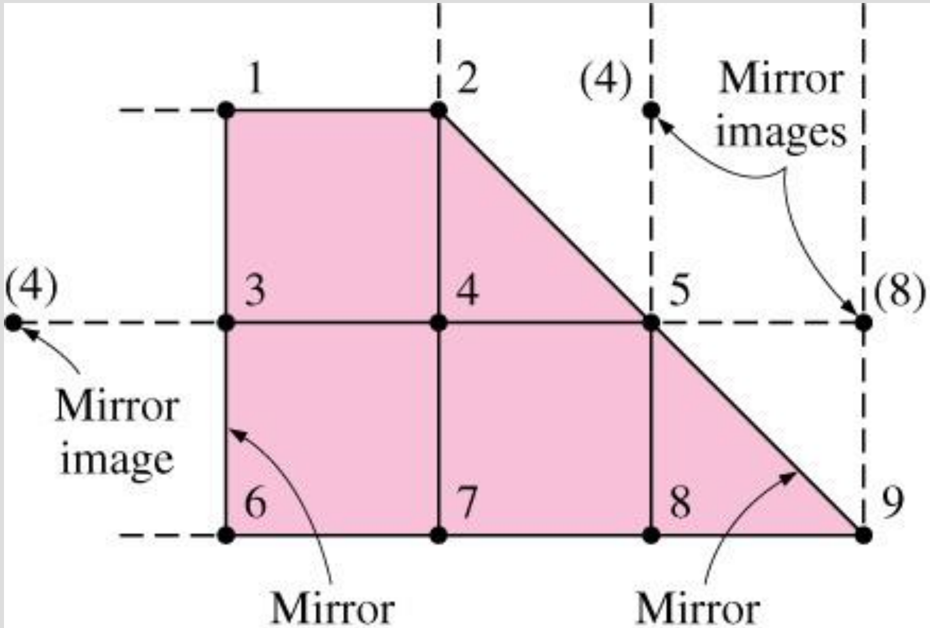


FIGURE 5–34

Converting the boundary nodes 3 and 5 on symmetry lines to interior nodes by using mirror images.

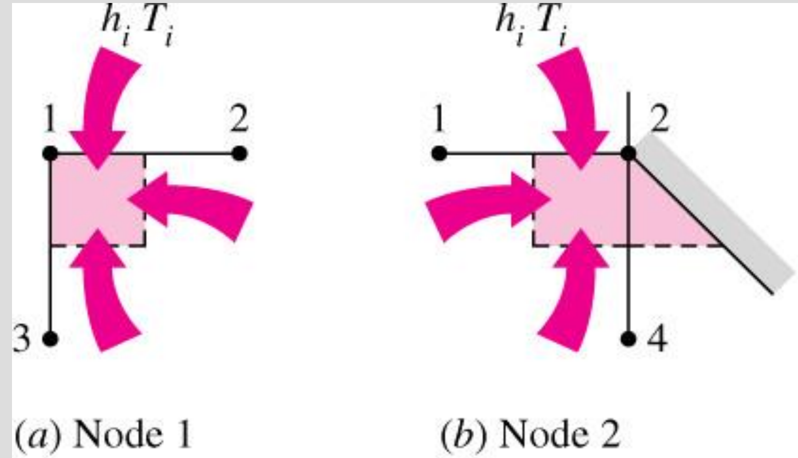


FIGURE 5–33

Schematics for energy balances on the volume elements of nodes 1 and 2.

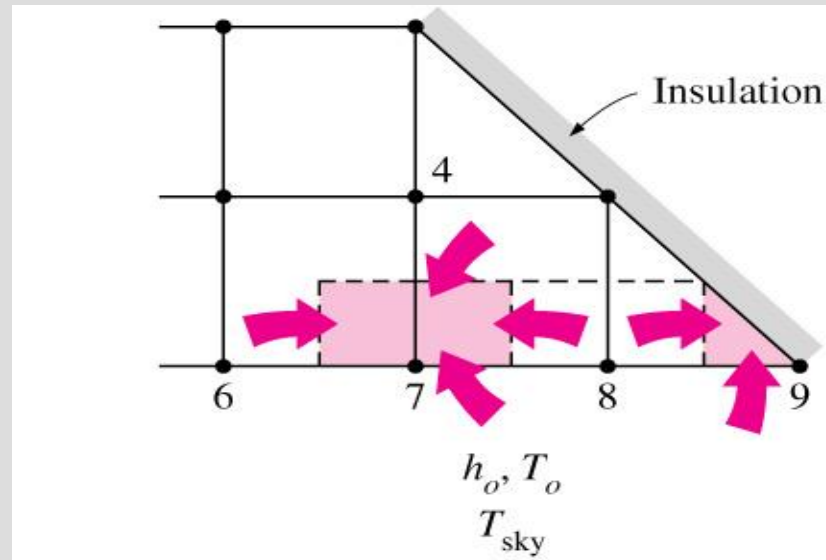


FIGURE 5–35

Schematics for energy balances on the volume elements of nodes 7 and 9.

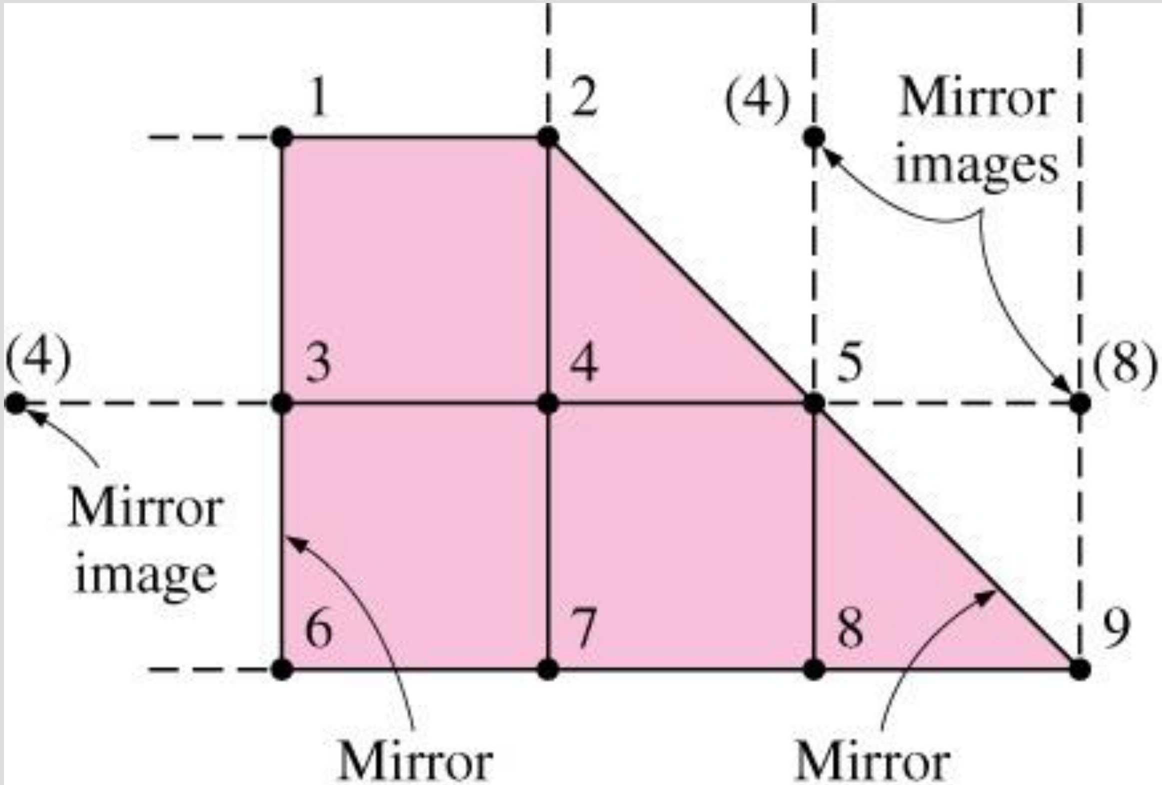


FIGURE 5–34

Converting the boundary nodes 3 and 5 on symmetry lines to interior nodes by using mirror images.

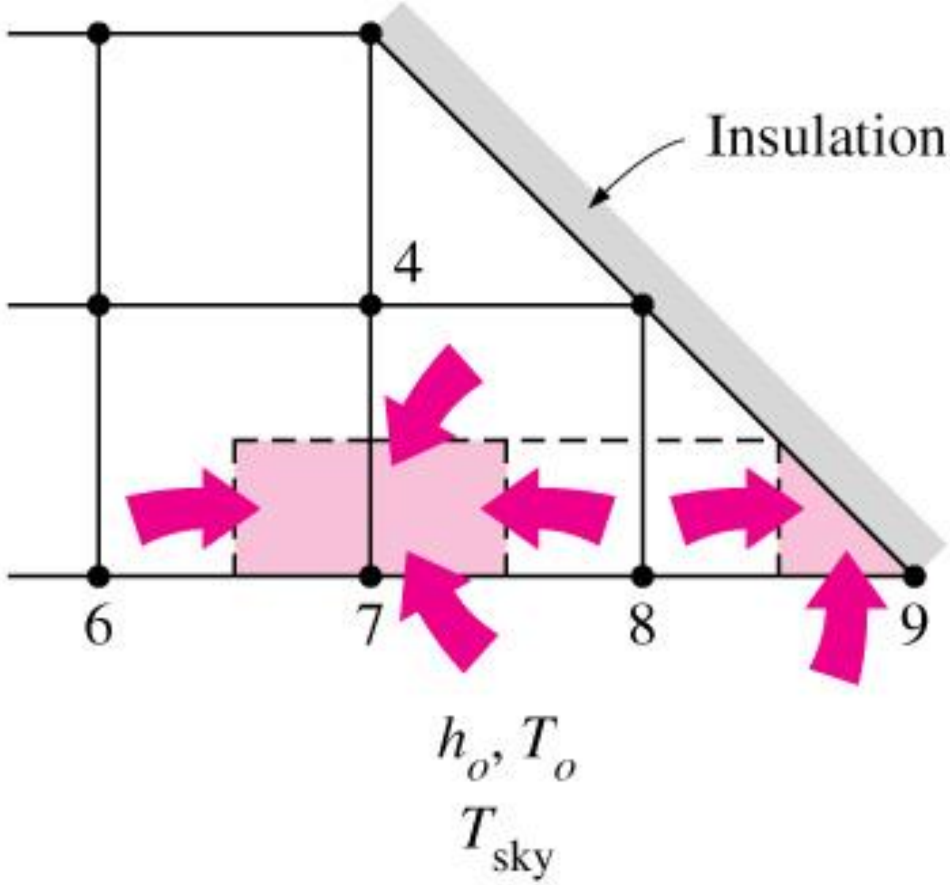


FIGURE 5–35

Schematics for energy balances on the volume elements of nodes 7 and 9.

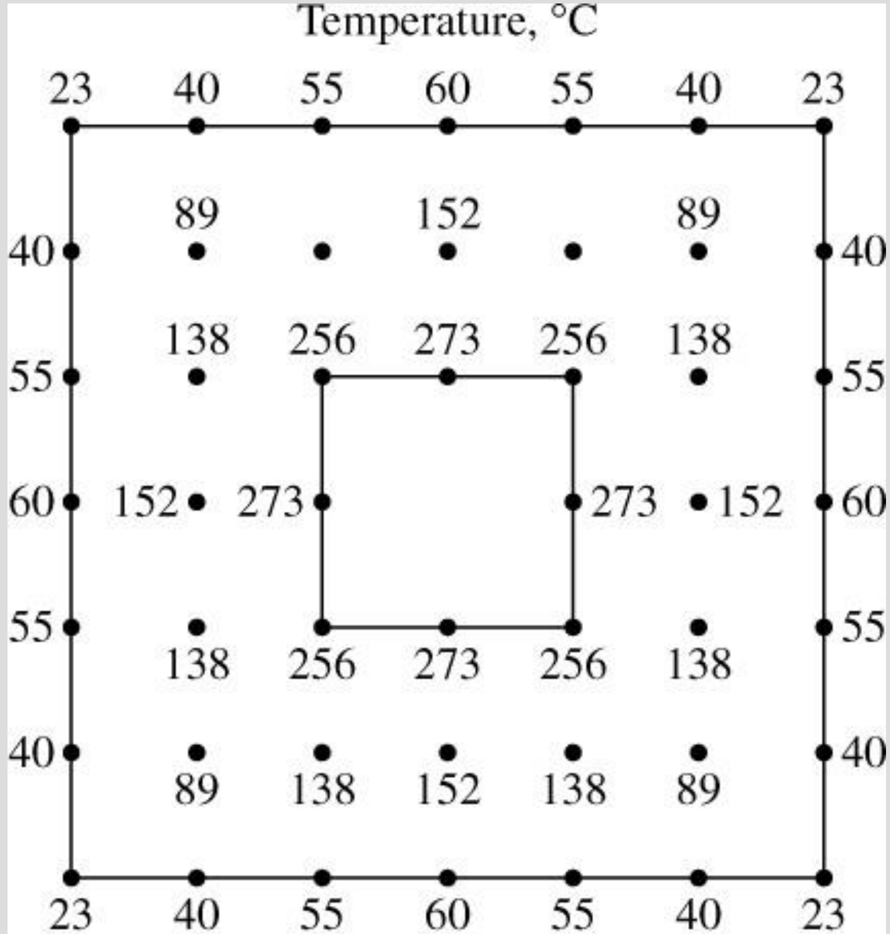


FIGURE 5–36

The variation of temperature in the chimney.

Ejercicio

5–55 Hot combustion gases of a furnace are flowing through a concrete chimney ($k = 1.4$ W/m · °C) of rectangular cross section. The flow section of the chimney is 20 cm × 40 cm, and the thickness of the wall is 10 cm. The average temperature of the hot gases in the chimney is $T_i = 280°C$, and the average convection heat transfer coefficient inside the chimney is $h_i = 75$ W/m² · °C. The chimney is losing heat from its outer surface to the ambient air at $T_o = 15°C$ by convection with a heat transfer coefficient of $h_o = 18$ W/m² · °C and to the sky by radiation. The emissivity of the outer surface of the wall is $\varepsilon = 0.9$, and the effective sky temperature is estimated to be 250 K. Using the finite difference method with $\Delta x = \Delta y = 10$ cm and taking full advantage of symmetry, (*a*) obtain the finite difference formulation of this problem for steady two-dimensional heat transfer, (*b*) determine the temperatures at the nodal points of a cross section, and (*c*) evaluate the rate of heat loss for a 1-m-long section of the chimney.